

DIAGONALIZATION OF TRANSFER MATRIX OF SUPERSYMMETRY $U_q(\widehat{sl}(M+1|N+1))$ CHAIN WITH A BOUNDARY

November 14, 2012

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Abstract

We study the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ analogue of the supersymmetric t - J model with a boundary, in the framework of the algebraic analysis method. We diagonalize the commuting transfer matrix by using the bosonization of the vertex operator associated with the quantum affine supersymmetry.

1 Introduction

There have been many developments in the exactly solvable models. Various methods were invented to solve models. The algebraic analysis method [1] provides a powerful method to study exactly solvable models. This paper is devoted to the algebraic analysis method to open boundary problem of exactly solvable model. In this paper we study the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ analogue of the supersymmetric t - J model with a boundary, where $M, N = 0, 1, 2, \dots$ such that $M \neq N$. The supersymmetric t - J model was proposed in an attempt to understand high-temperature superconductivity. In the framework of the quantum inverse scattering method [2, 3], the investigations of the supersymmetric t - J model and its $U_q(\widehat{sl}(M+1|N+1))$ analogue have been carried out in several papers [7, 8, 10, 11, 12, 13, 14]. In the framework of the algebraic analysis method [1, 18], the $U_q(\widehat{sl}(M+1|N+1))$ chain "without boundaries" has been studied in few papers [15, 16, 17]. In this paper we focus our attention on the boundary condition of the exactly solvable model. We study the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain "with a boundary" in the framework of the algebraic analysis method [1, 23]. We diagonalize the commuting

transfer matrix of the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary. Several solvable models with a boundary have been studied by means of algebraic analysis method [23, 24, 25, 26, 27, 28, 29]. Here we would like to draw reader's attention to new technical aspect in our problem. Generally speaking, in the algebraic analysis method, the transfer matrix $T_B(z)$ of the solvable model with a boundary is written by the product of the vertex operators $\Phi_j^*(z)$ and $\Phi_j(z)$ associated with the quantum affine symmetry

$$T_B^{(i)}(z) = g \sum_{j=1}^{M+N+2} \Phi_j^*(z^{-1}) K^{(i)}(z)_j^j \Phi_j(z) (-1)^{[v_j]}.$$

Here $K^{(i)}(z)_j^j$ is the matrix element of the boundary K -matrix. The key of solving the problem is the bosonization of the boundary state ${}_{(i)}\langle B|$ that satisfies the following condition

$${}_{(i)}\langle B| T_B^{(i)}(z) = {}_{(i)}\langle B|.$$

By using the bosonizations of the vertex operators, we construct the boundary state ${}_{(i)}\langle B|$. Our calculations depend heavily on the bosonization formulae of the vertex operators. For solvable models that are governed by the quantum symmetry $U_q(\widehat{sl}(N))$, $U_q(A_2^{(2)})$ or the elliptic symmetry $U_{q,p}(\widehat{sl}(N))$ [23, 24, 25, 26, 27, 28], the bosonizations of the vertex operators are realized by "monomial". However the bosonizations of the vertex operators for the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ are realized by "sum". For instance the bosonizations of the vertex operators $\Phi_{M+1+j}^*(z)$ ($j = 1, 2, \dots, N+1$) are written by the sum $\sum_{\epsilon_1, \epsilon_2, \dots, \epsilon_j}$ as followings (see (4.73)).

$$\begin{aligned} \Phi_{M+1+j}^*(z) &= \sum_{\epsilon_1, \epsilon_2, \dots, \epsilon_j = \pm} e^{\frac{M\pi\sqrt{-1}}{M-N}} q^{j-1} (q - q^{-1})^M (qz)^{-1} \prod_{k=1}^j \epsilon_k \prod_{k=1}^{M+j} \int_{C_{M+1+j}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \\ &\times \frac{1}{\prod_{k=0}^M (1 - qw_k/w_{k+1}) (1 - qw_{k+1}/w_k) \prod_{k=1}^{j-1} (1 - q^{\epsilon_k} w_{M+k}/w_{M+k+1})} \\ &\times \eta_0 \xi_0 : \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) X_{M+1, \epsilon_1}^-(qw_{M+1}) \cdots X_{M+j, \epsilon_j}^-(qw_{M+j}) : \eta_0 \xi_0. \end{aligned}$$

Technically this is cool part of our problem. Surprisingly we shall conclude that the bosonization of the boundary state ${}_{(i)}\langle B|$ is realized by "monomial", though those of the vertex operator is realized by "sum". The bosonization of the boundary state ${}_{(i)}\langle B|$ is constructed by acting a monomial of exponential $e^{G^{(i)}}$ on the highest weight vector $\langle \Lambda_{M+1} | \in V^*(\Lambda_{M+1})$ of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$.

$${}_{(i)}\langle B| = \langle \Lambda_{M+1} | e^{G^{(i)}}.$$

Here $G^{(i)}$ is quadratic in the bosonic operators (see (5.5)). We would like to give a comment on the earlier study in the framework of the algebraic analysis [29]. The supersymmetric t - J model with a boundary (the supersymmetry $U_q(\widehat{sl}(2|1))$ chain with a boundary) was studied and the bosonization conjecture of the boundary state was given in [29]. However, their conjecture of the boundary state is different from our bosonization upon the special case of $M = 1, N = 0$. In this paper we give not only the bosonization formulae of the boundary state, but also give complete proof that the vector ${}_{(i)}\langle B|$ becomes

the eigenvector of the transfer matrix $T_B^{(i)}(z)$. In this paper we classify the boundary K -matrix and find a new solution that has three different diagonal elements (see (B.11)). Of-course we construct the boundary state associated with this new K -matrix.

The text is organized as follows. In section 2 we introduce the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ analogue of finite supersymmetric finite t - J model with double boundaries. We introduce the $U_q(\widehat{sl}(M+1|N+1))$ analogue of semi-infinite t - J model as the limit of the finite chain. In section 3 we give mathematical formulation of the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary. This formulation is based on the representation theory of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$. This formulation is free from the difficulty of divergence. In section 4 we review the bosonizations of the vertex operators and give the integral representations of the vertex operators. In section 5 we give the bosonizations of the boundary state, that is the main result of this paper. We give complete proof of the bosonization of the boundary state. In appendix A we summarize the figures that we use in section 2. In appendix B we classify the diagonal solutions of the boundary Yang-Baxter equation associated with $U_q(\widehat{sl}(M+1|N+1))$. New solution is presented even for the small rank case $U_q(\widehat{sl}(2|1))$. In appendix C we summarize the normal orderings, that we use in section 4 and 6.

2 $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary

In this section we introduce the $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary. We fix a complex number $0 < |q| < 1$ and two natural numbers $M, N = 0, 1, 2, \dots$ such that $M \neq N$.

2.1 Finite $U_q(\widehat{sl}(M+1|N+1))$ chain

In this section we introduce the finite $U_q(\widehat{sl}(M+1|N+1))$ chain with double boundaries. We follow the general scheme given by [19, 20, 21, 14]. In what follows we use the standard notation of the q -integer

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}. \quad (2.1)$$

Let us introduce the signatures ν_i ($i = 1, 2, \dots, M+N+2$) by

$$\nu_1 = \nu_2 = \dots = \nu_{M+1} = +, \quad \nu_{M+2} = \nu_{M+3} = \dots = \nu_{M+N+2} = -. \quad (2.2)$$

Let us set the vector space $V = \oplus_{j=1}^{M+N+2} \mathbf{C} v_j$. The \mathbf{Z}_2 -grading of the basis $\{v_j\}$ of V is chosen to be $[v_j] = \frac{\nu_j+1}{2}$ ($j = 1, 2, \dots, M+N+2$).

Definition 2.1 We set the R -matrix $R(z) \in \text{End}(V \otimes V)$ associated with the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ as followings [7, 8].

$$R(z) = r(z)\bar{R}(z), \quad \bar{R}(z)v_{j_1} \otimes v_{j_2} = \sum_{k_1, k_2=1}^{M+N+2} v_{k_1} \otimes v_{k_2} \bar{R}(z)_{k_1, k_2}^{j_1, j_2}. \quad (2.3)$$

Here we have set

$$\bar{R}(z)_{j,j}^{j,j} = \begin{cases} -1 & (1 \leq j \leq M+1), \\ -\frac{(q^2 - z)}{(1 - q^2 z)} & (M+2 \leq j \leq M+N+2), \end{cases} \quad (2.4)$$

$$\bar{R}(z)_{i,j}^{i,j} = \frac{(1-z)q}{(1-q^2z)} \quad (1 \leq i \neq j \leq M+N+2), \quad (2.5)$$

$$\bar{R}(z)_{i,j}^{j,i} = \begin{cases} (-1)^{[v_i][v_j]} \frac{(1-q^2)}{(1-q^2z)} & (1 \leq i < j \leq M+N+2), \\ (-1)^{[v_i][v_j]} \frac{(1-q^2)z}{(1-q^2z)} & (1 \leq j < i \leq M+N+2), \end{cases} \quad (2.6)$$

$$\bar{R}(z)_{i,j}^{i,j} = 0 \quad \text{otherwise.} \quad (2.7)$$

Here we have set

$$r(z) = z^{\frac{1-M+N}{M-N}} \exp \left(- \sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{m[(M-N)m]_q} q^m (z^m - z^{-m}) \right). \quad (2.8)$$

For instance the function $r(z)$ is written as following in the case $M > N$.

$$r(z) = z^{\frac{1-M+N}{M-N}} \frac{(q^2z; q^{2(M-N)})_{\infty} (q^{2(M-N)}z^{-1}; q^{2(M-N)})_{\infty}}{(q^{2(M-N)}z; q^{2(M-N)})_{\infty} (q^2z^{-1}; q^{2(M-N)})_{\infty}}. \quad (2.9)$$

Here we have used the infinite product

$$(z; p)_{\infty} = \prod_{n=0}^{\infty} (1 - p^n z) \quad (|p| < 1).$$

Graphically, the R -matrix $R(z)$ is represented in Fig.1 in appendix A. The R -matrix $R(z)$ satisfies the graded Yang-Baxter equation.

$$R_{12}(z_1/z_2)R_{13}(z_1/z_3)R_{23}(z_2/z_3) = R_{23}(z_2/z_3)R_{13}(z_1/z_3)R_{12}(z_1/z_2), \quad (2.10)$$

where $R_{12}(z)$, $R_{13}(z)$ and $R_{23}(z)$ act in $V \otimes V \otimes V$ with $R_{12}(z) = R(z) \otimes 1$, $R_{23}(z) = 1 \otimes R(z)$, etc.

The relation (2.10), expressed in terms of matrix elements, can be rewritten in the following form.

$$\begin{aligned} & \sum_{j_1, j_2, j_3=1}^{M+N+2} R(z_1/z_2)_{i_1, i_2}^{j_1, j_2} R(z_1/z_3)_{j_1, i_3}^{k_1, j_3} R(z_2/z_3)_{j_2, j_3}^{k_2, k_3} (-1)^{([v_{j_1}]+[v_{k_1}])[v_{j_2}]} \\ &= \sum_{j_1, j_2, j_3=1}^{M+N+2} R(z_2/z_3)_{i_2, i_3}^{j_2, j_3} R(z_1/z_3)_{i_1, j_3}^{j_1, k_3} R(z_1/z_2)_{j_1, j_2}^{k_1, k_2} (-1)^{([v_{i_1}]+[v_{j_1}])[v_{j_2}]}. \end{aligned} \quad (2.11)$$

Multiplying the signature $(-1)^{[v_{k_1}][v_{k_2}]}$ to our R -matrix $R(z)_{k_1, k_2}^{j_1, j_2}$, we have the R -matrix $R^{PS}(z)$ of the Perk-Schultz model [8] : $R^{PS}(z)_{k_1, k_2}^{j_1, j_2} = (-1)^{[v_{k_1}][v_{k_2}]} R(z)_{k_1, k_2}^{j_1, j_2}$. The R -matrix $R^{PS}(z)$ of the Perk-Schultz model [8] satisfies the ungraded Yang-Baxter equation. The R -matrix $R(z)$ satisfies the initial condition $R(1) = P$ where P is the graded permutation operator : $P_{k_1, k_2}^{j_1, j_2} = \delta_{j_1, k_2} \delta_{j_2, k_1} (-1)^{[v_{k_1}][v_{k_2}]}$. The R -matrix $R(z)$ satisfies the unitary condition

$$R_{12}(z)R_{21}(1/z) = 1, \quad (2.12)$$

where $R_{21}(z) = PR_{12}(z)P$. The R -matrix $R(z)$ satisfies and the crossing symmetry

$$R_{12}^{-1, st_1}(z)M_1R_{12}^{st_1}(q^{2(N-M)}z)M_1^{-1} = 1. \quad (2.13)$$

Here we have set the matrix $M \in \text{End}(V)$ defined by

$$M_{i,j} = \delta_{i,j}M_j, \quad M_j = \begin{cases} q^{-2(j-1)} & (1 \leq j \leq M+1), \\ q^{-2(2M+2-j)} & (M+2 \leq j \leq M+N+2). \end{cases} \quad (2.14)$$

We have the commutativity $[M \otimes M, R(z)] = 0$. For instance the various supertranspositions of the R -matrix are given by

$$\begin{aligned} R^{st_1}(z)_{i,j}^{k,l} &= R(z)_{k,j}^{i,l} (-1)^{[v_i]([v_i]+[v_k])}, \quad R^{st_2}(z)_{i,j}^{k,l} = R(z)_{i,l}^{k,j} (-1)^{[v_j]([v_j]+[v_l])}, \\ R^{st_{12}}(z)_{i,j}^{k,l} &= (-1)^{([v_i]+[v_j])([v_i]+[v_j]+[v_k]+[v_l])} R(z)_{k,l}^{i,j} = R(z)_{k,l}^{i,j}. \end{aligned} \quad (2.15)$$

In appendix B we classify the boundary K -matrix that satisfies the graded boundary Yang-Baxter equation (2.28), and find new diagonal solution (see (B.11)).

Definition 2.2 We set the K -matrix $K^{(i)}(z) \in \text{End}(V)$ ($i = 1, 2, 3$) associated with the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ as followings.

$$K^{(i)}(z) = z^{-\frac{2M}{M-N}} \frac{\varphi^{(i)}(z)}{\varphi^{(i)}(z^{-1})} \bar{K}^{(i)}(z) \quad (i = 1, 2, 3), \quad (2.16)$$

$$\bar{K}^{(i)}(z)v_j = \sum_{k=1}^{M+N+2} v_k \bar{K}^{(i)}(z)_k^j, \quad (2.17)$$

$$\bar{K}^{(i)}(z)_k^j = \delta_{j,k} \bar{K}^{(i)}(z)_j^j. \quad (2.18)$$

The $\bar{K}^{(i)}(z)_j^j$ and $\varphi^{(i)}(z)$ are given by following CONDITION 1, 2, 3.

CONDITION 1 : $\bar{K}^{(1)}(z)_j^j$ are $\varphi^{(1)}(z)$ are defined by followings.

$$\bar{K}^{(1)}(z)_j^j = 1 \quad (j = 1, 2, \dots, M+N+2), \quad (2.19)$$

and

$$\begin{aligned} \varphi^{(1)}(z) &= \exp \left(\sum_{m=1}^{\infty} \frac{[2(N+1)m]_q}{m[2(M-N)m]_q} z^{2m} + \sum_{j=1}^M \sum_{m=1}^{\infty} \frac{[2(M-N-j)m]_q}{2m[2(M-N)m]_q} (1-q^{2m}) z^{2m} \right. \\ &\quad \left. + \sum_{j=M+2}^{M+N+1} \sum_{m=1}^{\infty} \frac{[2(-M-N-2-j)m]_q}{2m[2(M-N)m]_q} (1+q^{2m}) z^{2m} - \sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{2m[(M-N)m]_q} q^m z^{2m} \right). \end{aligned} \quad (2.20)$$

CONDITION 2 : $\bar{K}^{(2)}(z)_j^j$ and $\varphi^{(2)}(z)$ are defined by followings. We fix a natural number $L = 1, 2, \dots, M+N+1$ and a complex number $r \in \mathbf{C}$.

$$\bar{K}^{(2)}(z)_j^j = \begin{cases} 1 & (1 \leq j = k \leq L), \\ \frac{1-r/z}{1-rz} & (L+1 \leq j = k \leq M+N+2). \end{cases} \quad (2.21)$$

Condition 2.1 : For $L \leq M+1$ we set

$$\varphi^{(2)}(z) = \varphi^{(1)}(z) \times \exp \left(- \sum_{m=1}^{\infty} \frac{[(M-N-L)m]_q}{m[(M-N)m]_q} (rq^{-L}z)^m \right). \quad (2.22)$$

Condition 2.2 : For $M+2 \leq L \leq M+N+2$ we set

$$\varphi^{(2)}(z) = \varphi^{(1)}(z) \times \exp \left(- \sum_{m=1}^{\infty} \frac{[(-M-N-2+L)m]_q}{m[(M-N)m]_q} (rq^{L-2M-2}z)^m \right). \quad (2.23)$$

CONDITION 3 : $\bar{K}^{(3)}(z)_j^j$ and $\varphi^{(3)}(z)$ are defined by followings. We fix two natural numbers $L, K = 1, 2, \dots, M+N$ such that $L+K \leq M+N+1$. We fix a complex number $r \in \mathbf{C}$.

$$\bar{K}^{(3)}(z)_j^j = \begin{cases} 1 & (1 \leq j = k \leq L), \\ \frac{1-r/z}{1-rz} & (L+1 \leq j = k \leq L+K), \\ z^{-2} & (L+K+1 \leq j = k \leq M+N+2), \end{cases} \quad (2.24)$$

Condition 3.1 : For $L+K \leq M+1$ we set

$$\begin{aligned} \varphi^{(3)}(z) &= \varphi^{(1)}(z) \\ &\times \exp \left(\sum_{m=1}^{\infty} \left\{ \frac{[(-M+N+L)m]_q}{m[(N-M)m]_q} (rq^{-L}z)^m + \frac{[(-M+N+L+K)m]_q}{m[(M-N)m]_q} (q^{L-K}z/r)^m \right\} \right). \end{aligned} \quad (2.25)$$

Condition 3.2 : For $L \leq M+1 \leq L+K+1$ we set

$$\begin{aligned} \varphi^{(3)}(z) &= \varphi^{(1)}(z) \\ &\times \exp \left(\sum_{m=1}^{\infty} \left\{ \frac{[(-M+N+L)m]_q}{m[(M-N)m]_q} (rq^{-L}z)^m + \frac{[(M+N+L+2-L-K)m]_q}{m[(M-N)m]_q} (q^{3L+K-2M-2}z/r)^m \right\} \right). \end{aligned} \quad (2.26)$$

Condition 3.3 : For $M+1 \leq L-1$ we set

$$\begin{aligned} \varphi^{(3)}(z) &= \varphi^{(1)}(z) \\ &\times \exp \left(\sum_{m=1}^{\infty} \left\{ \frac{[(M+N+2-L)m]_q}{m[(M-N)m]_q} (rq^{L-2M-2}z)^m + \frac{[(M+N+L+2-L-K)m]_q}{m[(M-N)m]_q} (q^{L-2M-2}z/r)^m \right\} \right). \end{aligned} \quad (2.27)$$

In what follows we sometimes just write $K(z)$ by dropping the suffix "(i)" from $K^{(i)}(z)$. For classification of K -matrix, see appendix B and the references [9, 10, 11]. Graphically, the K -matrix $K(z)$ is represented in Fig.2 in appendix A. The K -matrix $K(z) \in \text{End}(V)$ satisfies the graded boundary Yang-Baxter equation

$$K_2(z_2)R_{21}(z_1z_2)K_1(z_1)R_{12}(z_1/z_2) = R_{21}(z_1/z_2)K_1(z_1)R_{12}(z_1z_2)K_2(z_2). \quad (2.28)$$

The relation (2.28), expressed in terms of matrix elements, can be rewritten in the form

$$\begin{aligned} &\sum_{j_1, j_2, k_1, k_2=1}^{M+N+2} K(z_2)_{i_2}^{j_2} R(z_1z_2)_{i_1, j_2}^{j_1, k_2} K(z_1)_{j_1}^{k_1} R(z_1/z_2)_{k_2, k_1}^{l_2, l_1} (-1)^{([v_{j_1}]+[v_{k_1}])[v_{j_2}]} \\ &= \sum_{j_1, j_2, k_1, k_2=1}^{M+N+2} R(z_1/z_2)_{i_1, i_2}^{j_1, j_2} K(z_1)_{j_1}^{k_1} R(z_1z_2)_{j_2, k_1}^{k_2, l_1} K(z_2)_{k_2}^{l_2} (-1)^{([v_{j_1}]+[v_{k_1}])[v_{k_2}]}. \end{aligned} \quad (2.29)$$

The K -matrix $K(z)$ satisfies $K(1) = 1$. The K -matrix $K(z)$ satisfies the boundary unitary condition

$$K(z)K(1/z) = 1. \quad (2.30)$$

We set the dual K -matrix $K^+(z) \in \text{End}(V)$ by

$$K^+(z) = K(1/q^{N-M}z)^{st}M. \quad (2.31)$$

Graphically, the dual K -matrix $K^+(z)$ is represented in Fig.2 in appendix A. The dual K -matrix $K^+(z)$ satisfies the dual graded boundary Yang-Baxter equation

$$\begin{aligned} & K_2^+(z_2)^{st_2} M_2^{-1} R_{2,1}(1/q^{2(M-N)} z_1 z_2) M_2 K_1^+(z_1)^{st_1} R_{12}(z_2/z_1) \\ &= R_{21}(z_2/z_1) K_1^+(z_1)^{st_1} M_1^{-1} R_{12}(1/q^{2(M-N)} z_1 z_2) M_1 K_2^+(z_2)^{st_2}. \end{aligned} \quad (2.32)$$

We set the monodromy matrix $\mathcal{T}(z)$ by

$$\mathcal{T}(z) = R_{01}(z) R_{02}(z) \cdots R_{0,P}(z) \in \text{End}(V_P \otimes \cdots \otimes V_1 \otimes V_0), \quad (2.33)$$

where V_j are copies of V .

Definition 2.3 We introduce the transfer matrix $T_B^{fin}(z)$ by

$$T_B^{fin}(z) = \text{str}_{V_0}(K^+(z) \mathcal{T}(z^{-1})^{-1} K(z) \mathcal{T}(z)), \quad (2.34)$$

where the supertrace is defined by $\text{str}(A) = \sum_j (-1)^{[v_j]} A_{j,j}$.

Graphically, the transfer matrix $T_B^{fin}(z)$ is represented in Fig.3 in appendix A.

Proposition 2.4 The transfer matrix $T_B^{fin}(z)$ form a commutative family.

$$[T_B^{fin}(z_1), T_B^{fin}(z_2)] = 0 \quad \text{for any } z_1, z_2. \quad (2.35)$$

The commutativity can be proved by using unitarity and cross-symmetry, boundary Yang-Baxter equation and dual boundary Yang-Baxter equation [20, 21, 14]. We set the Hamiltonian H_B^{fin} by

$$H_B^{fin} = \frac{d}{dz} T_B^{fin}(z)|_{z=1} = \sum_{j=1}^{P-1} h_{j,j+1} + \frac{1}{2} \frac{d}{dz} K_0(z)|_{z=1} + \frac{\text{str}_{V_0}(K_0^+(1) h_{0,P})}{\text{str}_{V_0}(K_0^+(1))}, \quad (2.36)$$

where $h_{j,j+1} = P_{j,j+1} \frac{d}{dz} R_{j,j+1}(z)|_{z=1}$.

2.2 Semi-infinite $U_q(\widehat{sl}(M+1|N+1))$ chain

In this section we introduce the semi-infinite $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary. We consider the Hamiltonian (2.36) in the semi-infinite limit.

$$H_B^{(i)} = \lim_{P \rightarrow \infty} H_B^{fin} = \lim_{P \rightarrow \infty} \frac{d}{dz} T_B^{fin}(z)|_{z=1} = \sum_{j=1}^{\infty} h_{j,j+1} + \frac{1}{2} \frac{d}{dz} K_0^{(i)}(z)|_{z=1}, \quad (2.37)$$

which acts formally on the left-infinite tensor product space.

$$\cdots \otimes V \otimes V \otimes V. \quad (2.38)$$

We would like to diagonalize the Hamiltonian $H_B^{(i)}$ in the semi-infinite limit. It is convenient to study the transfer matrix $\tilde{T}_B^{(i)}(z) = \lim_{P \rightarrow \infty} T_B^{fin}(z)$, including the spectral parameter z , instead of the Hamiltonian $H_B^{(i)}$. The transfer matrix $\tilde{T}_B^{(i)}(z)$ corresponding to the semi-infinite limit is depicted in Fig.4 in appendix A. By convention, the lattice sites in Fig.4 are numbered $k = 1, 2, 3, \dots$ from right to left. Fig.4 in

appendix A describes a semi-infinite two dimensional lattice, with alternating spectral parameters. The transfer matrix $\tilde{T}_B^{(i)}(z)$ is rewritten as follows.

$$\tilde{T}_B^{(i)}(z) = \sum_{j=1}^{M+N+2} \tilde{\Phi}_j^*(z^{-1}) K^{(i)}(z)_j^j \tilde{\Phi}_j(z) (-1)^{[v_j]}. \quad (2.39)$$

Here the vertex operator $\tilde{\Phi}_j(z)$ and the dual vertex operator $\tilde{\Phi}_j^*(z)$ are depicted in Fig.5 and Fig.6 in appendix A, respectively. The vertex operators $\tilde{\Phi}_j(z)$ and $\tilde{\Phi}_j^*(z)$ satisfy the following commutation relations.

$$\tilde{\Phi}_{j_2}(z_2) \tilde{\Phi}_{j_1}(z_1) = \sum_{k_1, k_2=1}^{M+N+2} R(z_1/z_2)_{j_1, j_2}^{k_1, k_2} \tilde{\Phi}_{k_1}(z_1) \tilde{\Phi}_{k_2}(z_2) (-1)^{[v_{j_1}][v_{j_2}]}, \quad (2.40)$$

$$\tilde{\Phi}_{j_2}^*(z_2) \tilde{\Phi}_{j_1}^*(z_1) = \sum_{k_1, k_2=1}^{M+N+2} R(z_1/z_2)_{k_1, k_2}^{j_1, j_2} \tilde{\Phi}_{k_1}^*(z_1) \tilde{\Phi}_{k_2}^*(z_2) (-1)^{[v_{j_1}][v_{j_2}]}, \quad (2.41)$$

$$\tilde{\Phi}_{j_2}(z_2) \tilde{\Phi}_{j_1}^*(z_1) = \sum_{k_1, k_2=1}^{M+N+2} R^{-1, st_1}(z_1/z_2)_{j_1, j_2}^{k_1, k_2} \tilde{\Phi}_{k_1}^*(z_1) \tilde{\Phi}_{k_2}(z_2) (-1)^{[v_{k_1}][v_{k_2}]}. \quad (2.42)$$

From the graded boundary Yang-Baxter relation (2.28) and the commutation relations of the vertex operators, we have the commutativity of the transfer matrix $\tilde{T}_B^{(i)}(z)$.

$$[\tilde{T}_B^{(i)}(z_1), \tilde{T}_B^{(i)}(z_2)] = 0 \quad \text{for any } z_1, z_2. \quad (2.43)$$

In order to diagonalize the transfer matrix $\tilde{T}_B^{(i)}(z)$, we follow the strategy that we call the algebraic analysis method.

3 Mathematical formulation

In this section we give mathematical formulation of the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary, that is free from the difficulty of divergence [1, 18, 23].

3.1 Quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$

In this section we review the definition of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ [4, 6]. The Cartan matrix of the affine superalgebra $\widehat{sl}(M+1|N+1)$ is given by

$$(A_{i,j})_{0 \leq i, j \leq M+N+1} = \begin{pmatrix} 0 & -1 & 0 & \cdots & & \cdots & 0 & 1 \\ -1 & 2 & -1 & \cdots & & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & & & & \cdots \\ & \cdots & \cdots & \cdots & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 0 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & & & & & \cdots & -2 & 1 \\ 1 & 0 & \cdots & & & & & \cdots & 1 & -2 \end{pmatrix}. \quad (3.1)$$

Here the diagonal part is $(A_{i,i})_{0 \leq i \leq M+N+1} = (0, \overbrace{2, \dots, 2}^M, 0, \overbrace{-2, \dots, -2}^N)$. Let us introduce orthonormal basis $\{\epsilon_j | j = 1, 2, \dots, M+N+2\}$ with the bilinear form $(\epsilon_i | \epsilon_j) = \nu_i \delta_{i,j}$, where the signature $\nu_i = \pm 1$ is given in (2.2). Define $\bar{\epsilon}_i = \epsilon_i - \nu_i \frac{1}{M-N} \sum_{j=1}^{M+N+2} \epsilon_j$. The classical simple roots $\bar{\alpha}_i$ ($i = 1, 2, \dots, M+N+1$) and the classical fundamental weights $\bar{\Lambda}_i$ ($i = 1, 2, \dots, M+N+1$) are defined by

$$\bar{\alpha}_i = \nu_i \epsilon_i - \nu_{i+1} \epsilon_{i+1}, \quad \bar{\Lambda}_i = \sum_{j=1}^i \bar{\epsilon}_j \quad (i = 1, 2, \dots, M+N+1). \quad (3.2)$$

Introduce the affine weight Λ_0 and the null root δ having $(\Lambda_0 | \epsilon_i) = (\delta | \epsilon_i) = 0$ for $i = 1, 2, \dots, M+N+2$ and $(\Lambda_0 | \lambda_0) = (\delta | \delta) = 0$, $(\Lambda_0 | \delta) = 1$. The affine roots α_i ($i = 0, 1, 2, \dots, M+N+1$) and the affine fundamental weights Λ_i ($i = 0, 1, 2, \dots, M+N+1$) are given by

$$\alpha_0 = \delta - \sum_{j=1}^{M+N+1} \alpha_j, \quad \alpha_i = \bar{\alpha}_i \quad (i = 1, 2, \dots, M+N+1), \quad (3.3)$$

$$\Lambda_0 = \Lambda_0, \quad \Lambda_i = \Lambda_0 + \bar{\Lambda}_i \quad (i = 1, 2, \dots, M+N+1). \quad (3.4)$$

Definition 3.1 *The quantum supersymmetry $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$ ($M, N = 0, 1, 2, \dots$, and $M \neq N$) is a q -analogue of the universal enveloping algebra $\widehat{\mathfrak{sl}}(M+1|N+1)$ generated by the Chevalley generators $\{e_i, f_i, h_i | i = 0, 1, 2, \dots, M+N+1\}$. The \mathbf{Z}_2 -grading of the generators are $[e_0] = [f_0] = [e_{M+1}] = [f_{M+1}] = 1$ and zero otherwise.*

The Cartan-Kac relations : For $i, j = 0, 1, \dots, M+N+1$, the generators subject to the following relations.

$$[h_i, h_j] = 0, \quad [h_i, e_j] = A_{i,j} e_j, \quad [h_i, f_j] = -A_{i,j} f_j, \quad [e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}. \quad (3.5)$$

For $i, j = 0, 1, \dots, M+N+1$ such that $|A_{i,j}| = 0$, the generators subject to the following relations.

$$[e_i, e_j] = 0, \quad [f_i, f_j] = 0. \quad (3.6)$$

The Serre relations : For $i, j = 0, 1, \dots, M+N+1$ such that $|A_{i,j}| = 1$ and $i \neq 0, M+1$, the generators subject to the following relations.

$$[e_i, [e_i, e_j]_{q^{-1}}]_q = 0, \quad [f_i, [f_i, f_j]_{q^{-1}}]_q = 0. \quad (3.7)$$

For $M+N \geq 2$, the Serre relations of the fourth degree hold.

$$[[[e_i, e_j]_q, e_k]_{q^{-1}}, e_j] = 0, \quad [[[[f_i, f_j]_q, f_k]_{q^{-1}}, f_j] = 0, \\ (i, j, k) = (M+N-1, 0, 1), (M-1, M, M+1). \quad (3.8)$$

For $(M, N) = (1, 0)$ the extra Serre relations of the fifth degree hold.

$$[e_0, [e_2, [e_0, [e_2, e_1]_q]]]_{q^{-1}} = [e_2, [e_0, [e_2, [e_0, e_1]_q]]]_{q^{-1}}, \quad (3.9)$$

$$[f_0, [f_2, [f_0, [f_2, f_1]_q]]]_{q^{-1}} = [f_2, [f_0, [f_2, [f_0, f_1]_q]]]_{q^{-1}}. \quad (3.10)$$

For $(M, N) = (0, 1)$ the extra Serre relations of the fifth degree hold.

$$[e_0, [e_1, [e_0, [e_1, e_2]_q]]]_{q^{-1}} = [e_1, [e_0, [e_1, [e_0, e_2]_q]]]_{q^{-1}}, \quad (3.11)$$

$$[f_0, [f_1, [f_0, [f_1, f_2]_q]]]_{q^{-1}} = [f_1, [f_0, [f_1, [f_0, f_2]_q]]]_{q^{-1}}. \quad (3.12)$$

Here and throughout this paper, we use the notations

$$[X, Y]_\xi = XY - (-1)^{[X][Y]} \xi YX. \quad (3.13)$$

We write $[X, Y]_1$ as $[X, Y]$ for simplicity. The quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ has the \mathbf{Z}_2 -graded Hopf-algebra structure. We take the following coproduct.

$$\Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \quad \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad (3.14)$$

and the antipode

$$S(e_i) = -q^{-h_i} e_i, \quad S(f_i) = -f_i q^{h_i}, \quad S(h_i) = -h_i. \quad (3.15)$$

The coproduct Δ satisfies an algebra automorphism $\Delta(XY) = \Delta(X)\Delta(Y)$ and the antipode S satisfies a \mathbf{Z}_2 -graded algebra anti-automorphism $S(XY) = (-1)^{[X][Y]} S(Y)S(X)$. The multiplication rule for the tensor product is \mathbf{Z}_2 -graded and is defined for homogeneous elements $X, Y, X', Y' \in U_q(\widehat{sl}(N|1))$ and $v \in V, w \in W$ by $X \otimes Y \cdot X' \otimes Y' = (-1)^{[Y][X']} XX' \otimes YY'$ and $X \otimes Y \cdot v \otimes w = (-1)^{[Y][v]} Xv \otimes Yw$, which extends to inhomogeneous elements through linearity.

3.2 Mathematical formulation

In this section we give mathematical formulation of our problem. We introduce the evaluation representation V_z of the $(M+N+2)$ dimensional basic representation $V = \oplus_{j=1}^{M+N+2} \mathbf{C} v_j$. Let $E_{i,j}$ be the $(M+N+2) \times (M+N+2)$ matrix whose (i,j) -elements is unity and zero elsewhere : $E_{i,j} v_k = \delta_{j,k} v_i$. For $i = 1, 2, \dots, M+N+2$, we set the evaluation representation V_z with the vectors $\{v_i \otimes z^n | i = 1, 2, \dots, M+N+2; n \in \mathbf{Z}\}$.

$$\begin{aligned} e_i &= E_{i,i+1}, \quad f_i = \nu_i E_{i+1,i}, \quad h_i = \nu_i E_{i,i} - \nu_{i+1} E_{i+1,i+1}, \\ e_0 &= -z E_{M+N+2,1}, \quad f_0 = z^{-1} E_{1,M+N+2}, \quad h_0 = -E_{1,1} - E_{M+N+2,M+N+2}. \end{aligned} \quad (3.16)$$

Let V_z^* the dual space of V_z with vectors $\{v_i^* \otimes z^n | i = 1, 2, \dots, M+N+2; n \in \mathbf{Z}\}$ such that $(v_i^* \otimes z^m | v_j \otimes z^n) = \delta_{i,j} \delta_{m+n,0}$. The $U_q(\widehat{sl}(M+1|N+1))$ -module structure is given by $(xv|w) = (v|(-1)^{|x||v|} S(x)w)$ for $v \in V_z^*, w \in V_z$ and we call the module as V_z^{*S} . For $i = 1, 2, \dots, M+N+2$, we have the explicit action on V_z^{*S} as follows.

$$\begin{aligned} e_i &= -\nu_i \nu_{i+1} q^{-\nu_i} E_{i+1,i}, \quad f_i = -\nu_i q^{\nu_i} E_{i,i+1}, \quad h_i = -\nu_i E_{i,i} + \nu_{i+1} E_{i+1,i+1}, \\ e_0 &= qz E_{1,M+N+2}, \quad f_0 = q^{-1} z^{-1} E_{M+N+2,1}, \quad h_0 = E_{1,1} + E_{M+N+2,M+N+2}. \end{aligned} \quad (3.17)$$

Definition 3.2 Let $V(\lambda)$ be the highest weight $U_q(\widehat{sl}(M+1|N+1))$ -module with the highest weight λ . We define the type-I vertex operators $\Phi(z)$ and $\Phi^*(z)$ as the intertwiners of $U_q(\widehat{sl}(M+1|N+1))$ -module if they exist.

$$\Phi(z) : V(\lambda) \rightarrow V(\mu) \otimes V_z, \quad \Phi^*(z) : V(\mu) \rightarrow V(\lambda) \otimes V_z^{*S}, \quad (3.18)$$

$$\Phi(z) \cdot x = \Delta(x) \cdot \Phi(z), \quad \Phi^*(z) \cdot x = \Delta(x) \cdot \Phi^*(z), \quad (3.19)$$

for $x \in U_q(\widehat{sl}(M+1|N+1))$.

We expand the vertex operators $\Phi(z) = \sum_{j=1}^{M+N+2} \Phi_j(z) \otimes v_j$ and $\Phi^*(z) = \sum_{j=1}^{M+N+2} \Phi_j^*(z) \otimes v_j^*$. The vertex operators $\Phi_j(z)$ and $\Phi_j^*(z)$ satisfy the following commutation relations.

$$\Phi_{j_2}(z_2)\Phi_{j_1}(z_1) = \sum_{k_1, k_2=1}^{M+N+2} R(z_1/z_2)_{j_1, j_2}^{k_1, k_2} \Phi_{k_1}(z_1)\Phi_{k_2}(z_2)(-1)^{[v_{j_1}][v_{j_2}]}, \quad (3.20)$$

$$\Phi_{j_2}^*(z_2)\Phi_{j_1}^*(z_1) = \sum_{k_1, k_2=1}^{M+N+2} R(z_1/z_2)_{k_1, k_2}^{j_1, j_2} \Phi_{k_1}^*(z_1)\Phi_{k_2}^*(z_2)(-1)^{[v_{j_1}][v_{j_2}]}, \quad (3.21)$$

$$\Phi_{j_2}(z_2)\Phi_{j_1}^*(z_1) = \sum_{k_1, k_2=1}^{M+N+2} R^{-1, st_1}(z_1/z_2)_{j_1, j_2}^{k_1, k_2} \Phi_{k_1}^*(z_1)\Phi_{k_2}(z_2)(-1)^{[v_{k_1}][v_{k_2}]}. \quad (3.22)$$

The vertex operators satisfy the inversion relations

$$\Phi_i(z)\Phi_j^*(z) = g^{-1}(-1)^{[v_i]}\delta_{i,j}, \quad \sum_{k=1}^{M+N+2} (-1)^{[v_k]}\Phi_k^*(z)\Phi_k(z) = g^{-1}. \quad (3.23)$$

Here we have used

$$g = e^{\frac{\pi\sqrt{-1}M}{2(M-N)}} \exp\left(-\sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{m[(M-N)m]_q} q^m\right). \quad (3.24)$$

Definition 3.3 We set the transfer matrix $T_B^{(i)}(z)$ by

$$T_B^{(i)}(z) = \sum_{j=1}^{M+N+2} \Phi_j^*(z^{-1})K^{(i)}(z)_j^j \Phi_j(z)(-1)^{[v_j]}. \quad (3.25)$$

From the graded boundary Yang-Baxter relation (2.28) and the commutation relations of the vertex operators, we have the commutativity of the transfer matrix $T_B^{(i)}(z)$.

Proposition 3.4 The transfer matrix $T_B^{(i)}(z)$ forms a commutative family.

$$[T_B^{(i)}(z_1), T_B^{(i)}(z_2)] = 0 \quad \text{for any } z_1, z_2. \quad (3.26)$$

Following the strategy proposed in [1, 23], we consider our problem upon the following identification.

$$T_B^{(i)}(z) = \tilde{T}_B^{(i)}(z), \quad \Phi_j(z) = \tilde{\Phi}_j(z), \quad \Phi_j^*(z) = \tilde{\Phi}_j^*(z). \quad (3.27)$$

The point of using the vertex operators $\Phi_j(z), \Phi_j^*(z)$ is that they are well-defined objects, free from the difficulty of divergence. Let us set $V(\lambda)^*$ the restricted dual module of the highest weight $V(\lambda)$.

Definition 3.5 We call the eigenvector ${}_{(i)}\langle B| \in V^*(\lambda)$ with eigenvalue 1 the boundary state.

$${}_{(i)}\langle B|T_B^{(i)}(z) = {}_{(i)}\langle B|. \quad (3.28)$$

We would like to construct the boundary state ${}_{(i)}\langle B| \in V^*(\lambda)$. Multiplying the vertex operator $\Phi_j^*(z)$ from the right and using the inversion relation (3.23), we have the following.

Proposition 3.6 The boundary state ${}_{(i)}\langle B|$ is characterized by

$${}_{(i)}\langle B|\Phi_j^*(z^{-1})K^{(i)}(z)_j^j = {}_{(i)}\langle B|\Phi_j^*(z) \quad (j = 1, 2, \dots, M+N+2). \quad (3.29)$$

In order to construct the boundary state ${}_{(i)}\langle B|$, it is convenient to introduce the bosonizations of the vertex operators $\Phi_j^*(z)$.

4 Vertex operator

In this section we review the bosonization of the vertex operators. We give the integral representation of the vertex operators, which are convenient for the construction of the boundary state ${}_{(i)}\langle B|$.

4.1 Drinfeld realization

In order to give the bosonizations, it is convenient to introduce the Drinfeld realization of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ [5, 6].

Definition 4.1 [6] *The generators of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$, which we call the Drinfeld generators, are given by*

$$X_{i,m}^{\pm}, h_{i,n}, h_i, c, \quad (i = 1, 2, \dots, M+N+1, m \in \mathbf{Z}, n \in \mathbf{Z}_{\neq 0}). \quad (4.1)$$

The \mathbf{Z}_2 -grading of the Drinfeld generators are $|X_m^{\pm, M+1}| = 1$ for $m \in \mathbf{Z}$ and zero otherwise. For $i, j = 1, 2, \dots, M+N+1$, the Drinfeld generators are subject to the following relations.

$$c : \text{central}, [h_i, h_{j,m}] = 0, \quad (4.2)$$

$$[h_{i,m}, h_{j,n}] = \frac{[A_{i,j}m]_q [cm]_q}{m} \delta_{m+n,0}, \quad (4.3)$$

$$[h_i, X_j^{\pm}(z)] = \pm A_{i,j} X_j^{\pm}(z), \quad (4.4)$$

$$[h_{i,m}, X_j^+(z)] = \frac{[A_{i,j}m]_q}{m} q^{-c|m|/2} z^m X_j^+(z), \quad (4.5)$$

$$[h_{i,m}, X_j^-(z)] = -\frac{[A_{i,j}m]_q}{m} q^{c|m|/2} z^m X_j^-(z), \quad (4.6)$$

$$(z_1 - q^{\pm A_{i,j}} z_2) X_i^{\pm}(z_1) X_j^{\pm}(z_2) = (q^{\pm A_{j,i}} z_1 - z_2) X_j^{\pm}(z_2) X_i^{\pm}(z_1), \quad \text{for } |A_{i,j}| \neq 0, \quad (4.7)$$

$$[X_i^{\pm}(z_1), X_j^{\pm}(z_2)] = 0, \quad \text{for } |A_{i,j}| = 0, \quad (4.8)$$

$$[X_i^+(z_1), X_j^-(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1}) z_1 z_2} (\delta(q^{-c} z_1/z_2) \psi_i^+(q^{\frac{c}{2}} z_2) - \delta(q^c z_1/z_2) \psi_i^-(q^{-\frac{c}{2}} z_2)), \quad (4.9)$$

$$(X_i^{\pm}(z_1) X_i^{\pm}(z_2) X_j^{\pm}(z) - (q + q^{-1}) X_i^{\pm}(z_1) X_j^{\pm}(z) X_i^{\pm}(z_2) + X_j^{\pm}(z) X_i^{\pm}(z_1) X_i^{\pm}(z_2)) \\ + (z_1 \leftrightarrow z_2) = 0, \quad \text{for } |A_{i,j}| = 1, i \neq M+1, \quad (4.10)$$

$$(X_{M+1}^{\pm}(z_1) X_{M+2}^{\pm}(w_1) X_{M+1}^{\pm}(z_2) X_M^{\pm}(w_2) - q^{-1} X_{M+1}^{\pm}(z_1) X_{M+2}^{\pm}(w_1) X_M^{\pm}(w_2) X_{M+1}^{\pm}(z_2) \\ - q X_{M+1}^{\pm}(z_1) X_{M+1}^{\pm}(z_2) X_M^{\pm}(w_2) X_{M+2}^{\pm}(w_1) + X_{M+1}^{\pm}(z_1) X_M^{\pm}(w_2) X_{M+1}^{\pm}(z_2) X_{M+2}^{\pm}(w_1) \\ + X_{M+2}^{\pm}(w_1) X_{M+1}^{\pm}(z_2) X_M^{\pm}(w_2) X_{M+1}^{\pm}(z_1) - q^{-1} X_{M+2}^{\pm}(w_1) X_M^{\pm}(w_2) X_{M+1}^{\pm}(z_2) X_{M+1}^{\pm}(z_1) \\ - q X_{M+1}^{\pm}(z_2) X_M^{\pm}(w_2) X_{M+2}^{\pm}(w_1) X_{M+1}^{\pm}(z_1) + X_M^{\pm}(w_2) X_{M+1}^{\pm}(z_2) X_{M+2}^{\pm}(w_1) X_{M+1}^{\pm}(z_1)) \\ + (z_1 \leftrightarrow z_2) = 0, \quad (4.11)$$

where we have used $\delta(z) = \sum_{m \in \mathbf{Z}} z^m$. Here we have set the generating functions

$$X_j^{\pm}(z) = \sum_{m \in \mathbf{Z}} X_{j,m}^{\pm} z^{-m-1}, \quad (4.12)$$

$$\psi_i^{\pm}(z) = q^{h_i} \exp \left((q - q^{-1}) \sum_{m=1}^{\infty} h_{i,m} z^{-m} \right), \quad (4.13)$$

$$\psi_i^-(z) = q^{-h_i} \exp \left(-(q - q^{-1}) \sum_{m=1}^{\infty} h_{i,-m} z^m \right). \quad (4.14)$$

The relation between the Chevalley generators and the Drinfeld generators are obtained by the followings.

$$e_i = X_{i,0}^+, \quad f_i = X_{i,0}^- \quad (i = 1, 2, \dots, M + N + 1), \quad (4.15)$$

$$h_0 = c - h_1 - h_2 - \dots - h_{M+N+1}, \quad (4.16)$$

$$\begin{aligned} e_0 &= (-1)^{N+1} [X_{M+N+1,0}^-, \dots, [X_{M+2,0}^-, [X_{M+1,0}^-, \dots, [X_{2,0}^-, X_{1,0}^-]_{q^{-1}} \dots]_{q^{-1}}]_q \dots]_q \\ &\times q^{-h_1 - h_2 - \dots - h_{M+N+1}}, \end{aligned} \quad (4.17)$$

$$\begin{aligned} f_0 &= q^{h_1 + h_2 + \dots + h_{M+N+1}} \\ &\times [\dots [\dots [X_{1,-1}^+, X_{2,0}^+]_q, \dots, X_{M+1,0}^+]_q, X_{M+2,0}^+]_{q^{-1}}, \dots, X_{M+N+1,0}^+]_{q^{-1}}. \end{aligned} \quad (4.18)$$

4.2 Bosonization

In this section we review the bosonizations of the Drinfeld realizations of the quantum supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ [15]. In what follows we assume the level $c = 1$, where we have the simplest realization. Let us introduce the bosons

$$a_n^i, b_n^j, c_n^j, Q_{a^i}, Q_{b^j}, Q_{c^j}, \quad (n \in \mathbf{Z}, i = 1, 2, \dots, M+1, j = 1, 2, \dots, N+1), \quad (4.19)$$

satisfying the following commutation relations.

$$[a_m^i, a_n^j] = \delta_{i,j} \delta_{m+n,0} \frac{[m]_q^2}{m}, \quad [a_0^i, Q_{a^j}] = \delta_{i,j}, \quad (4.20)$$

$$[b_m^i, b_n^j] = -\delta_{i,j} \delta_{m+n,0} \frac{[m]_q^2}{m}, \quad [b_0^i, Q_{b^j}] = -\delta_{i,j}, \quad (4.21)$$

$$[c_m^i, c_n^j] = \delta_{i,j} \delta_{m+n,0} \frac{[m]_q^2}{m}, \quad [c_0^i, Q_{c^j}] = \delta_{i,j}. \quad (4.22)$$

Other commutation relations vanish. We set $h_i = h_{i,0}$. For $i = 1, 2, \dots, M$ and $j = 1, 2, \dots, N$, we set

$$Q_{h_i} = Q_{a^i} - Q_{a^{i+1}}, \quad Q_{h_{M+1}} = Q_{a^{M+1}} + Q_{b^1}, \quad Q_{h_{M+1+j}} = -Q_{b^j} + Q_{b^{j+1}}. \quad (4.23)$$

It is convenient to introduce the generating function $h^i(z; \beta)$ by

$$h^i(z; \beta) = - \sum_{n \neq 0} \frac{h_{i,n}}{[n]_q} q^{-\beta|n|} z^{-n} + Q_{h_i} + h_{i,0} \log z \quad (\beta \in \mathbf{R}). \quad (4.24)$$

We introduce the q -difference operator defined by

$$\partial_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}. \quad (4.25)$$

In what follows we use the standard normal ordering $::$. For instance we set

$$: a_m^i a_n^i := \begin{cases} a_m^i a_n^i & (m < 0) \\ a_n^i a_m^i & (m > 0), \end{cases} \quad : a_0^i Q_{a^i} := Q_{a^i} a_0^i := Q_{a^i} a_0^i. \quad (4.26)$$

Theorem 4.2 [15] *The Drinfeld generators for the level $c = 1$ are realized as follows.*

$$c = 1, \quad (4.27)$$

$$h_{i,m} = a_m^i q^{-|m|/2} - a_m^{i+1} q^{|m|/2}, \quad (4.28)$$

$$h_{M+1,m} = a_m^{M+1} q^{-|m|/2} + b_m^1 q^{-|m|/2}, \quad (4.29)$$

$$h_{M+1+j,m} = -b_m^j q^{|m|/2} + b_m^{j+1} q^{-|m|/2}, \quad (4.30)$$

$$X_i^+(z) = : e^{h^i(z;1/2)} : e^{\pi\sqrt{-1}a_0^i}, \quad (4.31)$$

$$X_{M+1}^+(z) = : e^{h^{M+1}(z;1/2)} e^{c^1(z;0)} : \prod_{i=1}^M e^{-\pi\sqrt{-1}a_0^i}, \quad (4.32)$$

$$X_{M+1+j}^+(z) = : e^{h^{M+1+j}(z;1/2)} [\partial_z e^{-c^j(z;0)}] e^{c^{j+1}(z;0)} :, \quad (4.33)$$

$$X_i^-(z) = - : e^{-h^i(z;-1/2)} : e^{-\pi\sqrt{-1}a_0^i}, \quad (4.34)$$

$$X_{M+1}^-(z) = : e^{-h^{M+1}(z;-1/2)} [\partial_z e^{-c^1(z;0)}] : \prod_{i=1}^M e^{\pi\sqrt{-1}a_0^i}, \quad (4.35)$$

$$X_{M+1+j}^-(z) = - : e^{-h^{M+1+j}(z;-1/2)} e^{c^j(z;0)} [\partial_z e^{-c^{j+1}(z;0)}] :, \quad (4.36)$$

for $i = 1, 2, \dots, M$ and $j = 1, 2, \dots, N$.

4.3 Highest weight module

In this section we study the space that the bosonizations act. We introduce the vacuum vector $|0\rangle$ by

$$a_n^i |0\rangle = b_n^j |0\rangle = c_n^j |0\rangle \quad (n \leq 0, i = 1, 2, \dots, M+1, j = 1, 2, \dots, N+1). \quad (4.37)$$

For $\lambda_a^i, \lambda_b^j, \lambda_c^j \in \mathbf{C}$ ($i = 1, \dots, M+1, j = 1, \dots, N+1$), we set the vector

$$|\lambda_a^1, \dots, \lambda_a^{M+1}, \lambda_b^1, \dots, \lambda_b^{N+1}, \lambda_c^1, \dots, \lambda_c^{N+1}\rangle = e^{\sum_{i=1}^{M+1} \lambda_a^i Q_{a^i} + \sum_{j=1}^{N+1} \lambda_b^j Q_{b^j} + \sum_{j=1}^{N+1} \lambda_c^j Q_{c^j}} |0\rangle. \quad (4.38)$$

The Fock module $\mathcal{F}_{\lambda_a^1, \dots, \lambda_a^{M+1}, \lambda_b^1, \dots, \lambda_b^{N+1}, \lambda_c^1, \dots, \lambda_c^{N+1}}$ is generated by acting creation operators $a_{-n}^i, b_{-n}^j, c_{-n}^j$ ($n > 0$) over the vector $|\lambda_a^1, \dots, \lambda_a^{M+1}, \lambda_b^1, \dots, \lambda_b^{N+1}, \lambda_c^1, \dots, \lambda_c^{N+1}\rangle$. In order to obtain the highest weight vectors of $U_q(\widehat{sl}(M+1|N+1))$, we impose the conditions.

$$\begin{aligned} h_i |\lambda_a^1, \dots, \lambda_a^{M+1}, \lambda_b^1, \dots, \lambda_b^{N+1}, \lambda_c^1, \dots, \lambda_c^{N+1}\rangle &= \lambda_i |\lambda_a^1, \dots, \lambda_a^{M+1}, \lambda_b^1, \dots, \lambda_b^{N+1}, \lambda_c^1, \dots, \lambda_c^{N+1}\rangle, \\ e_i |\lambda_a^1, \dots, \lambda_a^{M+1}, \lambda_b^1, \dots, \lambda_b^{N+1}, \lambda_c^1, \dots, \lambda_c^{N+1}\rangle &= 0 \quad (i = 1, 2, \dots, M+N+1). \end{aligned}$$

Solving these equations, we have two classes of solutions.

(1) $|\Lambda_i\rangle$ ($i = 1, 2, \dots, M+N+1$).

For $i = 1, 2, \dots, M+1$, we identify

$$|\Lambda_i\rangle = |\overbrace{\beta+1, \dots, \beta+1}^i, \overbrace{\beta, \dots, \beta}^{M+N+2-i}, \overbrace{0, \dots, 0}^{N+1}\rangle \quad (\beta \in \mathbf{C}). \quad (4.39)$$

For $j = 1, \dots, N+1$, we identify

$$|\Lambda_{M+1+j}\rangle = |\overbrace{\beta+1, \dots, \beta+1}^{M+1+j}, \overbrace{\beta, \dots, \beta}^{N+1-j}, \overbrace{0, \dots, 0}^j, \overbrace{-1, \dots, -1}^{N+1-j}\rangle \quad (\beta \in \mathbf{C}). \quad (4.40)$$

(2) $|(1-\alpha)\Lambda_0 + \alpha\Lambda_{M+1}\rangle$ for $\alpha \in \mathbf{C}$. We identify

$$|(1-\alpha)\Lambda_0 + \alpha\Lambda_{M+1}\rangle = |\overbrace{\beta, \dots, \beta}^{M+1}, \overbrace{\beta - \alpha, \dots, \beta - \alpha}^{N+1}, \overbrace{-\alpha, \dots, -\alpha}^{N+1}\rangle \quad (\beta \in \mathbf{C}). \quad (4.41)$$

For $i = 1, 2, \dots, M+1$, $j = 1, 2, \dots, N+1$ and $\alpha, \beta \in \mathbf{C}$, we set the spaces.

$$\mathcal{F}_{(\Lambda_i, \beta)} = \bigoplus_{i_1, \dots, i_{M+N+1} \in \mathbf{Z}} \mathcal{F}_{(\overbrace{\beta+1, \dots, \beta+1}^i, \overbrace{\beta, \dots, \beta}^{M+N+2-i}, \overbrace{0, \dots, 0}^{N+1}) \circ (i_1, i_2, \dots, i_{M+N+1})}, \quad (4.42)$$

$$\mathcal{F}_{(\Lambda_{M+1+j}, \beta)} = \bigoplus_{i_1, \dots, i_{M+N+1} \in \mathbf{Z}} \mathcal{F}_{(\overbrace{\beta+1, \dots, \beta+1}^{M+1+j}, \overbrace{\beta, \dots, \beta}^{N+1-j}, \overbrace{0, \dots, 0}^j, \overbrace{-1, \dots, -1}^{N+1-j}) \circ (i_1, i_2, \dots, i_{M+N+1})} \quad (4.43)$$

$$\mathcal{F}_{(\alpha, \beta)} = \bigoplus_{i_1, \dots, i_{M+N+1} \in \mathbf{Z}} \mathcal{F}_{(\overbrace{\beta, \dots, \beta}^{M+1}, \overbrace{\beta - \alpha, \dots, \beta - \alpha}^{N+1}, \overbrace{-\alpha, \dots, -\alpha}^{N+1}) \circ (i_1, i_2, \dots, i_{M+N+1})}. \quad (4.44)$$

Here we have used the following abbreviation.

$$\begin{aligned} & (\lambda_a^1, \dots, \lambda_a^{M+1}, \lambda_b^1, \dots, \lambda_b^{N+1}, \lambda_c^1, \dots, \lambda_c^{N+1}) \circ (i_1, i_2, \dots, i_{M+N+1}) \\ &= (\lambda_a^1, \dots, \lambda_a^{M+1}, \lambda_b^1, \dots, \lambda_b^{N+1}, \lambda_c^1, \dots, \lambda_c^{N+1}) \\ &+ (i_1, i_2 - i_1, \dots, i_{M+1} - i_M, i_{M+1} - i_{M+2}, \dots, i_{M+N} - i_{M+N+1}, i_{M+N+1} \\ & \quad , i_{M+1} - i_{M+2}, \dots, i_{M+N} - i_{M+N+1}, i_{M+N+1}). \end{aligned} \quad (4.45)$$

The actions of $U_q(\widehat{sl}(M+1|N+1))$ on the spaces $\mathcal{F}_{(\Lambda_i, \beta)}, \mathcal{F}_{(\alpha, \beta)}$ are closed. However, these modules are not irreducible in general. In order to obtain irreducible module, we introduce ξ - η system. We introduce the operators ξ_m^j and η_m^j ($j = 1, 2, \dots, N+1; m \in \mathbf{Z}$) by

$$\xi^j(z) = \sum_{m \in \mathbf{Z}} \xi_m^j z^{-m} =: e^{-c^j(z)} :, \quad \eta^j(z) = \sum_{m \in \mathbf{Z}} \eta_m^j z^{-m-1} =: e^{c^j(z)} :. \quad (4.46)$$

The Fourier components $\xi_m^j = \oint \frac{dz}{2\pi\sqrt{-1}} z^{m-1} \xi^j(z)$ and $\eta_m^j = \oint \frac{dz}{2\pi\sqrt{-1}} z^m \eta^j(z)$ are well-defined on the spaces $\mathcal{F}_{(\Lambda_i, \beta)}, \mathcal{F}_{(\alpha, \beta)}$ for $\alpha \in \mathbf{Z}$. They satisfy the anti-commutation relations.

$$\{\xi_m^j, \eta_n^j\} = \delta_{m+n, 0}, \quad \{\xi_m^j, \xi_n^j\} = \{\eta_m^j, \eta_n^j\} = 0 \quad (j = 1, 2, \dots, N+1). \quad (4.47)$$

Here we have used $\{a, b\} = ab + ba$. They commute with each other.

$$[\xi_m^j, \eta_n^{j'}] = [\xi_m^j, \xi_n^{j'}] = [\eta_m^j, \eta_n^{j'}] = 0 \quad (1 \leq j \neq j' \leq N+1). \quad (4.48)$$

We focus our attention on the operators η_0^j, ξ_0^j satisfying $(\eta_0^j)^2 = 0, (\xi_0^j)^2 = 0$. They satisfy

$$\text{Im}(\eta_0^j) = \text{Ker}(\eta_0^j), \quad \text{Im}(\xi_0^j) = \text{Ker}(\xi_0^j). \quad (4.49)$$

The products $\eta_0^j \xi_0^j$ and $\xi_0^j \eta_0^j$ are projection operators, which satisfy

$$\eta_0^j \xi_0^j + \xi_0^j \eta_0^j = 1, \quad (4.50)$$

$$(\eta_0^j \xi_0^j)^2 = \eta_0^j \xi_0^j, \quad (\xi_0^j \eta_0^j)^2 = \xi_0^j \eta_0^j, \quad (\eta_0^j \xi_0^j)(\xi_0^j \eta_0^j) = 0, \quad (\xi_0^j \eta_0^j)(\eta_0^j \xi_0^j) = 0. \quad (4.51)$$

Hence we have a direct sum decomposition for $i = 1, 2, \dots, M + N + 1, j = 1, 2, \dots, N + 1$.

$$\mathcal{F}_{(\Lambda_i, \beta)} = \eta_0^j \xi_0^j \mathcal{F}_{(\Lambda_i, \beta)} \oplus \xi_0^j \eta_0^j \mathcal{F}_{(\Lambda_i, \beta)}, \quad \mathcal{F}_{(\alpha, \beta)} = \eta_0^j \xi_0^j \mathcal{F}_{(\alpha, \beta)} \oplus \xi_0^j \eta_0^j \mathcal{F}_{(\alpha, \beta)}, \quad (4.52)$$

and

$$\text{Ker}(\eta_0^j) = \eta_0^j \xi_0^j \mathcal{F}_{(\Lambda_i, \beta)} \quad \text{or} \quad \eta_0^j \xi_0^j \mathcal{F}_{(\alpha, \beta)}, \quad \text{Coker}(\eta_0^j) = \xi_0^j \eta_0^j \mathcal{F}_{(\Lambda_i, \beta)} \quad \text{or} \quad \xi_0^j \eta_0^j \mathcal{F}_{(\alpha, \beta)}. \quad (4.53)$$

We set the operators η_0 and ξ_0 by

$$\eta_0 = \prod_{j=1}^{N+1} \eta_0^j, \quad \xi_0 = \prod_{j=1}^{N+1} \xi_0^j. \quad (4.54)$$

Following the conjectures in [15], we expect the following identifications.

$$V(\Lambda_i) = \text{Coker} \eta_0 = \xi_0 \eta_0 \mathcal{F}_{(\Lambda_i, \beta)} \quad (i = 1, 2, \dots, M + N + 1), \quad (4.55)$$

$$V((1 - \alpha)\Lambda_0 + \alpha\Lambda_{M+1}) = \begin{cases} \text{Coker}(\eta_0) = \xi_0 \eta_0 \mathcal{F}_{(\alpha, \beta)} & (\alpha = 0, 1, 2, \dots), \\ \text{Ker}(\eta_0) = \eta_0 \xi_0 \mathcal{F}_{(\alpha, \beta)} & (\alpha = -1, -2, \dots). \end{cases} \quad (4.56)$$

Since the operators η_0^j and ξ_0^j commute with $U_q(\widehat{sl}(M+1|N+1))$ up to sign \pm , we can regard $\text{Ker}(\eta_0)$ and $\text{Coker}(\eta_0)$ as $U_q(\widehat{sl}(M+1|N+1))$ -module. In what follows we will work on the space, that is expected to be the irreducible highest weight module $V(\Lambda_{M+1})$.

$$V(\Lambda_{M+1}) = \xi_0 \eta_0 \mathcal{F}_{(1, \beta)}.$$

4.4 Vertex operator

In this section we give the bosonization of the vertex operators $\Phi_j^*(z)$, and give the integral representations of them. We set the following combinations of the Drinfeld generators.

$$h_{i,m}^* = \sum_{j=1}^{M+N+1} \frac{[\alpha_{i,j}m]_q [\beta_{i,j}m]_q}{[(M-N)m]_q [m]_q} h_{j,m}, \quad (4.57)$$

$$Q_{h_i^*} = \sum_{j=1}^{M+N+1} \frac{\alpha_{i,j} \beta_{i,j}}{M-N} Q_{h_{j,0}}, \quad h_{i,0}^* = \sum_{j=1}^{M+N+1} \frac{\alpha_{i,j} \beta_{i,j}}{M-N} h_{j,0}. \quad (4.58)$$

Here we have set

$$\alpha_{i,j} = \begin{cases} \text{Min}(i, j) & (\text{Min}(i, j) \leq M + 1), \\ 2(M + 1) - \text{Min}(i, j) & (\text{Min}(i, j) > M + 1), \end{cases} \quad (4.59)$$

$$\beta_{i,j} = \begin{cases} M - N - \text{Max}(i, j) & (\text{Max}(i, j) \leq M + 1), \\ -M - N - 2 + \text{Max}(i, j) & (\text{Max}(i, j) > M + 1). \end{cases} \quad (4.60)$$

We have the following commutation relations.

$$[h_{i,m}^*, h_{j,n}] = \delta_{i,j} \delta_{m+n,0} \frac{[m]_q^2}{m}, \quad [h_{i,m}^*, h_{j,m}^*] = \delta_{m+n,0} \frac{[\alpha_{i,j}m]_q [\beta_{i,j}m]_q [m]_q}{m[(M-N)m]_q}, \quad (4.61)$$

$$[h_{i,0}^*, Q_{h_j}] = \delta_{i,j}, \quad [h_{i,0}^*, Q_{h_j^*}] = \frac{\alpha_{i,j} \beta_{i,j}}{(M-N)}. \quad (4.62)$$

Theorem 4.3 [15] *The bosonic operator $\phi^*(z)$ given below satisfies the same commutation relations as the vertex operator $\Phi^*(z)$. In other words, the bosonizations of the vertex operator $\Phi^*(z)$ on the space $\mathcal{F}_{(\alpha,\beta)}$, $\mathcal{F}_{(\Lambda_i,\beta)}$, $\mathcal{F}_{(\Lambda_{M+1+j},\beta)}$ are given by the followings.*

$$\phi^*(z) = \sum_{j=1}^{M+N+2} \phi_j^*(z) \otimes v_j^*. \quad (4.63)$$

Here the bosonic operators $\phi_j^*(z)$ ($j = 1, 2, \dots, M+N+2$) are defined iteratively by

$$\phi_1^*(z) =: e^{h^*(qz; -1/2)} : \prod_{k=1}^{M+1} e^{\pi\sqrt{-1}\frac{k-1}{M-N}a_0^k}, \quad (4.64)$$

$$\nu_j q^{\nu_j} \phi_{j+1}^*(z) = -[\phi_j^*(z), f_j]_{q^{\nu_j}} \quad (j = 1, 2, \dots, M+N+1). \quad (4.65)$$

The \mathbf{Z}_2 -grading is given by $|\phi_j^*(z)| = \frac{\nu_j+1}{2}$ ($j = 1, 2, \dots, M+N+2$).

Corollary 4.4 *The bosonizations of the vertex operators $\Phi_j^*(z)$ on the space $\xi_0\eta_0\mathcal{F}_{(\alpha,\beta)}$ ($\alpha = 0, 1, 2, \dots$) are given by the following projection.*

$$\Phi_j^*(z) = \xi_0\eta_0 \cdot \phi_j^*(z) \cdot \xi_0\eta_0 \quad (j = 1, 2, \dots, M+N+2). \quad (4.66)$$

In what follows we call the bosonic operators $\phi_j^*(z)$ the vertex operators. We prepare the auxiliary operators $X_{M+j,\epsilon}^-(w)$ ($\epsilon = \pm$) by

$$X_{M+j}^-(w) = \frac{1}{(q - q^{-1})w} (X_{M+j,+}^-(w) - X_{M+j,-}^-(w)) \quad (j = 1, \dots, N+1). \quad (4.67)$$

In other words, we set

$$X_{M+1,\epsilon}^-(w) = : e^{-h^{M+1}(w) - c^1(q^\epsilon w)} : \prod_{j=1}^M e^{\pi\sqrt{-1}a_0^j} \quad (\epsilon = \pm), \quad (4.68)$$

$$X_{M+1+j,\epsilon}^-(w) = - : e^{-h^{M+j+1}(w) + c^j(w) - c^{j+1}(q^\epsilon w)} : \quad (\epsilon = \pm, j = 2, 3, \dots, N+1). \quad (4.69)$$

Using the normal orderings in appendix C we have the following normal orderings for $j = 1, 2, \dots, M$.

$$\begin{aligned} & : \phi_1^*(z) X_1^-(qw_1) \cdots X_j^-(qw_j) : X_{j+1}^-(qw_{j+1}) = e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qw_j(1 - qw_{j+1}/w_j)} ::, \\ & X_{j+1}^-(qw_{j+1}) : \phi_1^*(z) X_1^-(qw_1) \cdots X_j^-(qw_j) := -e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qw_{j+1}(1 - qw_j/w_{j+1})} ::, \\ & : \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) : X_{M+1}^-(qw_{M+1}) = -e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qw_M(1 - qw_{M+1}/w_M)} ::, \\ & X_{M+1}^-(qw_{M+1}) : \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) := -e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qw_{M+1}(1 - qw_M/w_{M+1})} ::. \end{aligned}$$

For $\epsilon = \pm$ and $j = 1, 2, \dots, N+1$, we have

$$\begin{aligned} & qX_{M+j,+}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) - X_{M+j+1,\epsilon}^-(w_2)X_{M+j,+}^-(w_1) \\ &= \frac{(q^2 - 1)}{(1 - qw_1/w_2)} : X_{M+j,+}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) :, \\ & qX_{M+j,-}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) - X_{M+j+1,\epsilon}^-(w_2)X_{M+j,-}^-(w_1) \\ &= \frac{(q^2 - 1)}{(1 - w_1/qw_2)} : X_{M+j,-}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) :. \end{aligned}$$

Using these normal orderings and (4.65), we have the following integral representations.

Proposition 4.5 *The vertex operators $\phi_j^*(z)$ ($j = 1, 2, \dots, M + N + 2$) have the following integral representations.*

$$\phi_1^*(z) = : e^{h^{*1}(qz)} : \prod_{k=1}^{M+1} e^{\pi\sqrt{-1}\frac{k-1}{M}a_0^k}, \quad (4.70)$$

$$\begin{aligned} \phi_i^*(z) &= e^{\frac{\pi\sqrt{-1}}{M}(i-1)}(q - q^{-1})^{i-1} \prod_{k=1}^{i-1} \int_{C_i} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{z^{-1}w_{i-1}}{\prod_{k=0}^{i-2} (1 - qw_k/w_{k+1})(1 - qw_{k+1}/w_k)} \\ &\times : \phi_1^*(z) X_1^-(qw_1) \cdots X_{i-1}^-(qw_{i-1}) : \quad (i = 2, \dots, M + 1), \end{aligned} \quad (4.71)$$

$$\begin{aligned} \phi_{M+2}^*(z) &= e^{\frac{M\pi\sqrt{-1}}{M-N}}(q - q^{-1})^M(qz)^{-1} \sum_{\epsilon=\pm} \epsilon \prod_{k=1}^{M+1} \int_{C_{M+2}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{1}{\prod_{k=0}^M (1 - qw_k/w_{k+1})(1 - qw_{k+1}/w_k)} \\ &\times : \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) X_{M+1,\epsilon}^-(qw_{M+1}) :, \end{aligned} \quad (4.72)$$

$$\begin{aligned} \phi_{M+1+j}^*(z) &= e^{\frac{M\pi\sqrt{-1}}{M-N}} q^{j-1} (q - q^{-1})^M (qz)^{-1} \sum_{\epsilon_1, \dots, \epsilon_j = \pm} \prod_{k=1}^j \epsilon_k \prod_{k=1}^{M+j} \int_{C_{M+1+j}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \\ &\times \frac{1}{\prod_{k=0}^M (1 - qw_k/w_{k+1})(1 - qw_{k+1}/w_k) \prod_{k=1}^{j-1} (1 - q^{\epsilon_k} w_{M+k}/w_{M+k+1})} \\ &\times : \phi_1^*(z) X_1^-(qw_1) \cdots X_M^-(qw_M) X_{M+1,\epsilon_1}^-(qw_{M+1}) \cdots X_{M+j,\epsilon_j}^-(qw_{M+j}) :, \\ &\quad (j = 1, 2, \dots, N + 1). \end{aligned} \quad (4.73)$$

Here we have read $w_0 = z$. We take the integration contour C_i ($i = 1, 2, \dots, M + N + 2$) to be simple closed curve that encircles $w_l = 0, qw_{l-1}$ but not $q^{-1}w_{l-1}$ for $l = 1, 2, \dots, i - 1$.

5 Boundary state

In this section we give the bosonization of the boundary state ${}_{(i)}\langle B|$. The construction of the boundary state is the main result of this paper. We give complete proof of this bosonization of the boundary state.

5.1 Boundary state

In this section we give the bosonization of the boundary state ${}_{(i)}\langle B|$. We use the highest weight vector $\langle \Lambda_{M+1} | \in V^*(\Lambda_{M+1})$ given by

$$\langle \Lambda_{M+1} | = \langle 0 | e^{-\beta \sum_{i=1}^{M+1} Q_{a^i} + (1-\beta) \sum_{j=1}^{N+1} Q_{b^j} + \sum_{j=1}^{N+1} Q_{c^j}}, \quad (5.1)$$

where $\langle 0 |$ is the vacuum vector satisfying

$$\langle 0 | a_n^i = \langle 0 | b_n^j = \langle 0 | c_n^j = 0 \quad (n \geq 0, i = 1, 2, \dots, M + 1, j = 1, 2, \dots, N + 1). \quad (5.2)$$

We have

$$\langle \Lambda_{M+1} | h_i = \delta_{i,M+1} \langle \Lambda_{M+1} |, \quad \langle \Lambda_{M+1} | c_0^j = -\langle \Lambda_{M+1} |, \quad (5.3)$$

for $i = 1, 2, \dots, M + N + 1$ and $j = 1, 2, \dots, N + 1$. In what follows we use the auxiliary function

$$\theta_m = \begin{cases} 1 & (m : \text{even}), \\ 0 & (m : \text{odd}). \end{cases} \quad (5.4)$$

Definition 5.1 We define the bosonic operators $G^{(i)}$ ($i = 1, 2, 3$) by

$$\begin{aligned} G^{(i)} &= -\frac{1}{2} \sum_{j=1}^{M+N+1} \sum_{m=1}^{\infty} \frac{mq^{-2m}}{[m]_q^2} h_m^j h_m^{*j} - \sum_{j=1}^{N+1} \sum_{m=1}^{\infty} \frac{mq^{-2m}}{[m]_q^2} c_m^j c_m^j \\ &+ \sum_{j=1}^{M+N+1} \sum_{m=1}^{\infty} \beta_{j,m}^{(i)} h_m^{*j} + \sum_{j=1}^{N+1} \sum_{m=1}^{\infty} \gamma_{j,m} c_m^j. \end{aligned} \quad (5.5)$$

Here we have set

$$\gamma_{j,m} = -\frac{q^{-m}}{[m]_q} \theta_m \quad (j = 1, 2, \dots, N + 1). \quad (5.6)$$

Here we have set $\beta_{j,m}^{(i)}$ ($i = 1, 2, 3$) as followings.

CONDITION 1 : For $i = 1$ we have set

$$\beta_{j,m}^{(1)} = \begin{cases} \frac{q^{-3m/2} - q^{-m/2}}{[m]_q} \theta_m & (1 \leq j \leq M), \\ \frac{-2q^{-3m/2}}{[m]_q} \theta_m & (j = M + 1), \\ \frac{q^{-3m/2} + q^{-m/2}}{[m]_q} \theta_m & (M + 2 \leq j \leq M + N + 1). \end{cases} \quad (5.7)$$

CONDITION 2 : For $i = 2$ we have set

$$\beta_{j,m}^{(2)} = \beta_{j,m}^{(1)} - \frac{r^m q^{(\alpha_L - 3/2)m}}{[m]_q} \delta_{j,L} \quad (j = 1, 2, \dots, M + N + 1). \quad (5.8)$$

Condition 2.1 : For $L \leq M + 1$ we have set

$$\alpha_L = -L. \quad (5.9)$$

Condition 2.2 : For $M + 2 \leq L \leq M + N + 1$ we have set

$$\alpha_L = L - 2M - 2. \quad (5.10)$$

CONDITION 3 : For $i = 3$ we have set

$$\beta_{j,m}^{(2)} = \beta_{j,m}^{(1)} - \frac{r^m q^{(\alpha_L - 3/2)m}}{[m]_q} \delta_{j,L} - \frac{q^{(\alpha_{L+K} - 3/2)m} / r^m}{[m]_q} \delta_{j,L+K} \quad (j = 1, 2, \dots, M + N + 1). \quad (5.11)$$

Condition 3.1 : For $L + K \leq M + 1$ we have set

$$(\alpha_L, \alpha_{L+K}) = (-L, L - K). \quad (5.12)$$

Condition 3.2 : For $L \leq M + 1 \leq L + K - 1$ we have set

$$(\alpha_L, \alpha_{L+K}) = (-L, 3L + K - 2M - 2). \quad (5.13)$$

Condition 3.3 : For $M + 2 \leq L$ we have set

$$(\alpha_L, \alpha_{L+K}) = (L - 2M - 2, 2M + K - L + 2). \quad (5.14)$$

The following is **main theorem** of this paper.

Theorem 5.2 *The bosonization of the boundary state ${}_{(i)}\langle B|$ ($i = 1, 2, 3$) is given by*

$${}_{(i)}\langle B| = \langle \Lambda_{M+1}| e^{G^{(i)}}. \quad (5.15)$$

Here the bosonic operator $G^{(i)}$ ($i = 1, 2, 3$) is given by (5.5). In other words the vector ${}_{(i)}\langle B|$ becomes the eigenvector of the transfer matrix $T_B^{(i)}(z)$ with the eigenvalue 1.

$${}_{(i)}\langle B| T_B^{(i)}(z) = {}_{(i)}\langle B|. \quad (5.16)$$

5.2 Excitation

In this section we introduce other eigenvectors of $T_B^{(i)}(z)$, that describes the excitations.

Definition 5.3 *We define the type-II vertex operators $\Psi(z)$ and $\Psi^*(z)$ as the intertwiners of $U_q(\widehat{sl}(M+1|N+1))$ -module if they exist.*

$$\Psi(z) : V(\lambda) \rightarrow V_z \otimes V(\mu), \quad \Psi^*(z) : V(\mu) \rightarrow V_z^{*S} \otimes V(\lambda), \quad (5.17)$$

$$\Psi(z) \cdot x = \Delta(x) \cdot \Psi(z), \quad \Psi^*(z) \cdot x = \Delta(x) \cdot \Psi^*(z), \quad (5.18)$$

for $x \in U_q(\widehat{sl}(M+1|N+1))$.

We expand the vertex operators $\Psi(z) = \sum_{j=1}^{M+N+2} v_j \otimes \Psi_j(z)$ and $\Psi^*(z) = \sum_{j=1}^{M+N+2} v_j^* \otimes \Psi_j^*(z)$. The type-II vertex operator $\Psi_\mu^*(\xi)$ and type-I vertex operators $\Phi_j(z)$, $\Phi_j^*(z)$ satisfy the following commutation relations.

$$\Psi_\mu^*(\xi) \Phi_j(z) = \tau(\xi/z) \Phi_j(z) \Psi_\mu^*(\xi) (-1)^{[v_\mu][v_j]}, \quad (5.19)$$

$$\Psi_\mu^*(\xi) \Phi_j^*(z) = \tau(\xi/z) \Phi_j^*(z) \Psi_\mu^*(\xi) (-1)^{[v_\mu][v_j]}. \quad (5.20)$$

Here we have set

$$\tau(z) = -z^{\frac{1-M+N}{M-N}} \exp \left(- \sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{m[(M-N)m]_q} (z^m - z^{-m}) \right). \quad (5.21)$$

Definition 5.4 *We call the following vectors ${}_{\mu_1, \mu_2, \dots, \mu_n(i)}\langle \xi_1, \xi_2, \dots, \xi_n|$ the excitations. We set*

$${}_{\mu_1, \mu_2, \dots, \mu_n(i)}\langle \xi_1, \xi_2, \dots, \xi_n| = {}_{(i)}\langle B| \Psi_{\mu_1}^*(\xi_1) \Psi_{\mu_2}^*(\xi_2) \cdots \Psi_{\mu_n}^*(\xi_n),$$

for $\mu_1, \mu_2, \dots, \mu_n = 1, 2, \dots, M+N+2$.

Corollary 5.5 *The excitations become the eigenvector of the transfer matrix $T_B^{(i)}(z)$.*

$${}_{\mu_1, \mu_2, \dots, \mu_n(i)}\langle \xi_1, \xi_2, \dots, \xi_n| T_B^{(i)}(z) = {}_{\mu_1, \mu_2, \dots, \mu_n(i)}\langle \xi_1, \xi_2, \dots, \xi_n| \prod_{\mu=1}^n \tau(\xi_\mu z) \tau(\xi_\mu / z). \quad (5.22)$$

We expect that the excitations (5.22) are the basis of the space of the physical state of the supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary.

6 Proof of main theorem

In this section we give complete proof of the main theorem. We would like to show

$${}_{(i)}\langle B|\phi_j^*(z^{-1})K^{(i)}(z)_j^j = {}_{(i)}\langle B|\phi_j^*(z^{-1}) \quad (j = 1, 2, \dots, M + N + 2). \quad (6.1)$$

It is convenient to use the following abbreviations.

$$h_+^i(z) = - \sum_{m=1}^{\infty} \frac{h_{i,m}}{[m]_q} q^{\frac{m}{2}} z^{-m}, \quad h_-^i(z) = \sum_{m=1}^{\infty} \frac{h_{i,-m}}{[m]_q} q^{\frac{m}{2}} z^m \quad (i = 1, 2, \dots, M + N + 1), \quad (6.2)$$

$$h_+^{*1}(z) = - \sum_{m=1}^{\infty} \frac{h_{1,m}^*}{[m]_q} q^{\frac{m}{2}} z^{-m}, \quad h_-^{*1}(z) = \sum_{m=1}^{\infty} \frac{h_{1,-m}^*}{[m]_q} q^{\frac{m}{2}} z^m, \quad (6.3)$$

$$c_+^j(z) = - \sum_{m=1}^{\infty} \frac{c_m^j}{[m]_q} z^{-m}, \quad c_-^j(z) = \sum_{m=1}^{\infty} \frac{c_{-m}^j}{[m]_q} z^m \quad (j = 1, 2, \dots, N + 1). \quad (6.4)$$

We set the function $D(z, w)$ by

$$D(z, w) = (1 - qzw)(1 - qz/w)(1 - qw/z)(1 - q/wz). \quad (6.5)$$

The function $D(z, w)$ is invariant under $(z, w) \rightarrow (1/z, w), (z, 1/w), (1/z, 1/w)$.

Proposition 6.1 *The operator $G^{(i)}$ ($i = 1, 2, 3$) given in (5.5) satisfies*

$$e^{G^{(i)}} h_{j,-m} e^{-G^{(i)}} = h_{j,-m} - q^{-2m} h_{j,m} + \frac{[m]_q^2}{m} \beta_{j,m}^{(i)} \quad (m > 0, j = 1, 2, \dots, M + N + 1), \quad (6.6)$$

$$e^{G^{(i)}} c_{-m}^j e^{-G^{(i)}} = c_{-m}^j - q^{-2m} c_m^j + \frac{[m]_q^2}{m} \gamma_{j,m} \quad (m > 0, j = 1, 2, \dots, N + 1). \quad (6.7)$$

Proposition 6.2 *For the boundary conditions $i = 1, 2, 3$, we have*

$${}_{(i)}\langle B|h_j = \delta_{j,M+1(i)} {}_{(i)}\langle B| \quad (j = 1, 2, \dots, M + N + 1), \quad (6.8)$$

$${}_{(i)}\langle B|h_{1,0}^* = -\frac{N+1}{M-N} {}_{(i)}\langle B|, \quad (6.9)$$

$${}_{(i)}\langle B|c_0^j = -{}_{(i)}\langle B| \quad (j = 1, 2, \dots, N + 1). \quad (6.10)$$

We show the relation (6.1) for each boundary condition ${}_{(i)}\langle B|$ ($i = 1, 2, 3$), case by case.

6.1 Boundary condition 1

In this section we show (6.1) for the boundary condition ${}_{(1)}\langle B|$. Very explicitly we would like to show

$$z^{\frac{M}{M-N}} \varphi^{(1)}(z^{-1}) {}_{(1)}\langle B|\phi_j^*(z) = z^{-\frac{M}{M-N}} \varphi^{(1)}(z) {}_{(1)}\langle B|\phi_j^*(z^{-1}) \quad (1 \leq j \leq M + N + 2). \quad (6.11)$$

We would like to comment that RHS (resp.LHS) is obtained from LHS (resp.RHS) under $z \rightarrow 1/z$. The proof for $1 \leq j \leq M + 1$ is similar as those of non-super $\widehat{sl}(N)$ case. The proof for $M + 2 \leq j \leq M + N + 2$ is different from those of non-super case. We prepare propositions.

Proposition 6.3 *The actions of $h_+^i(w)$, $h_-^{*1}(w)$ and $c_-^j(w)$ on the boundary state ${}_{(1)}\langle B|$ are given as followings.*

$${}_{(1)}\langle B|e^{-h_-^i(qw)} = g_i^{(1)}(w){}_{(1)}\langle B|e^{-h_+^i(q/w)} \quad (i = 1, 2, \dots, M + N + 1), \quad (6.12)$$

$${}_{(1)}\langle B|e^{h_-^{*1}(qw)} = \varphi^{(1)}(w){}_{(1)}\langle B|e^{h_+^{*1}(q/w)}, \quad (6.13)$$

$${}_{(1)}\langle B|e^{-c_-^j(qw)} = c_j^{(1)}(w){}_{(1)}\langle B|e^{-c_+^j(q/w)} \quad (j = 1, 2, \dots, N + 1). \quad (6.14)$$

Here we have set

$$g_i^{(1)}(w) = \begin{cases} (1 - w^2) & (1 \leq i \leq M), \\ (1 + w^2) & (i = M + 1), \\ 1 & (M + 2 \leq i \leq M + N + 1). \end{cases} \quad (6.15)$$

The function $\varphi^{(1)}(w)$ is given in (2.20) and $c_j^{(1)}(w) = 1$ for $j = 1, 2, \dots, N + 1$.

Proposition 6.4 *The following relation holds.*

$$\sum_{\epsilon=\pm} \int_C \frac{dw_1}{w_1} \frac{\epsilon q^\epsilon (1 - qw_1 w_2) e^{-c_+^j(q^{1+\epsilon} w_1) - c_+^j(q^{1-\epsilon}/w_1)}}{(1 - q^\epsilon w_1 w_2)(1 - q^\epsilon w_1/w_2)} = \int_C \frac{dw_1}{w_1} \frac{(-1 + w_1^2) e^{-c_+^j(w_1) - c_+^j(q^2/w_1)}}{(1 - w_1 w_2/q)(1 - w_1/qw_2)}. \quad (6.16)$$

Here the integration contour C encircles $w_1 = 0, qw_2^{\pm 1}$ not but $w_1 = q^{-1}w_2^{\pm 1}$. This integral is invariant under $w_2 \rightarrow 1/w_2$.

Proof for boundary condition 1. We show main theorem for the boundary condition ${}_{(1)}\langle B|$. We show the relation (6.11).

- The case for $j = 1$: ${}_{(1)}\langle B|\phi_1^*(z)$.

Using the bosonization (4.70) and the relation (6.13), we get LHS of (6.11) as following :

$$z^{\frac{M}{M-N}} \varphi^{(1)}(1/z) {}_{(1)}\langle B|\phi_1^*(z) = q^{-\frac{M}{M-N}} \varphi^{(1)}(z) \varphi^{(1)}(1/z) {}_{(1)}\langle B|e^{h_+^{*1}(qz) + h_+^{*1}(q/z)} e^{Q_{h_1^*}}. \quad (6.17)$$

This is invariant under $z \rightarrow 1/z$. Hence LHS and RHS of (6.11) coincide.

- The case for $j = 2$: ${}_{(1)}\langle B|\phi_2^*(z)$.

Using the bosonization (4.71), the relations (6.12), (6.13), and the normal orderings in appendix C, we get LHS of (6.11) as following :

$$\begin{aligned} & z^{\frac{M}{M-N}} \varphi^{(1)}(1/z) {}_{(1)}\langle B|\phi_2^*(z) \\ &= q^{-\frac{M}{M-N}-1} (q - q^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}} \varphi^{(1)}(z) \varphi^{(1)}(1/z) \\ &\times \int_{C_1} \frac{dw}{2\pi\sqrt{-1}w} \frac{(1 - w^2)(1 - q/zw)}{D(z, w)} {}_{(1)}\langle B|e^{Q_{h_1^* - h_1}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - h_+^1(qw) + h_+^1(q/w)}. \end{aligned} \quad (6.18)$$

We note that the integrand $\frac{\varphi^{(1)}(z)\varphi^{(1)}(1/z)}{D(z, w)} e^{h_+^{*1}(qz) + h_+^{*1}(q/z)}$ is invariant $z \rightarrow 1/z$. We have LHS–RHS of (6.11) as following :

$$\begin{aligned} & q^{-\frac{M}{M-N}} (q - q^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}} \varphi^{(1)}(z) \varphi^{(1)}(1/z) (z - z^{-1}) \\ &\times \int_{\tilde{C}_1} \frac{dw}{2\pi\sqrt{-1}w} \frac{(w^{-1} - w)}{D(z, w)} {}_{(1)}\langle B|e^{Q_{h_1^* - h_1}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - h_+^1(qw) - h_+^1(q/w)}. \end{aligned} \quad (6.19)$$

Here the integration contour \tilde{C}_1 encircles $w = 0, qz^{\pm 1}$ but not $w = q^{-1}z^{\pm 1}$. The integration contour \tilde{C}_1 is invariant under $w \rightarrow 1/w$. The integrand $\frac{(w^{-1}-w)}{D(z,w)}e^{-h_+^1(qw)-h_+^1(q/w)}$ creates just signature (-1) under $w \rightarrow 1/w$. Hence we have LHS-RHS=0.

- The case for $3 \leq j \leq M+1$: ${}_{(1)}\langle B|\phi_j^*(z)\rangle$.

Using the bosonization (4.71), the relations (6.12), (6.13) and normal orderings in appendix C, we have LHS-RHS of (6.11) as following.

$$\begin{aligned}
& z^{\frac{M}{M-N}}\varphi^{(1)}(1/z) {}_{(1)}\langle B|\phi_j^*(z) - z^{-\frac{M}{M-N}}\varphi^{(1)}(z) {}_{(1)}\langle B|\phi_j^*(1/z) \\
&= q^{-\frac{M}{M-N}}(q - q^{-1})^{j-1}\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z - z^{-1})e^{\frac{\pi\sqrt{-1}(j-1)}{M-N}} \\
&\times \prod_{k=1}^{j-1} \int_{\tilde{C}_j} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{j-2} (1 - q/w_k w_{k+1})}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} (w_1^{-1} - w_1) \prod_{k=2}^{j-1} (1 - w_k^2) \\
&\times {}_{(1)}\langle B|e^{Q_{h_1^*-h_1}\dots-h_{j-1}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{j-1}(h_+^k(qw_k)+h_+^k(q/w_k))}. \tag{6.20}
\end{aligned}$$

Here the integration contour \tilde{C}_j encircles $w_k = 0, qw_k^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq j-1$. Let's study the changing of the variable $w_1 \rightarrow 1/w_1$. We note that the integrand $\frac{1}{D(z,w_1)}e^{-h_+^1(qw_1)-h_+^1(q/w_1)}$ and the integration contour \tilde{C}_j are invariant under $w_1 \rightarrow 1/w_1$. Taking into account of symmetrization $\oint_{|w_1|=1} \frac{dw_1}{w_1} f(w_1) = \frac{1}{2} \oint_{|w_1|=1} \frac{dw_1}{w_1} (f(w_1) + f(1/w_1))$ and relation $(1 - q/w_1 w_2) - (w_1 \leftrightarrow 1/w_1) = (w_1^{-1} - w_1)(-q/w_2)$, we symmetrize the variables w_1, w_2, \dots, w_{j-2} , iteratively, we have

$$\begin{aligned}
& (-q/2)^{j-2} q^{-\frac{M}{M-N}}(q - q^{-1})^{j-1}\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z - z^{-1})e^{\frac{\pi\sqrt{-1}(j-1)}{M-N}} \\
&\times \prod_{k=1}^{j-1} \int_{\tilde{C}_j} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{j-2} (w_k^{-1} - w_k)^2 (w_{j-1}^{-1} - w_{j-1})}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} \\
&\times {}_{(i)}\langle B|e^{Q_{h_1^*-h_1}\dots-h_{j-1}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{j-1}(h_+^k(qw_k)+h_+^k(q/w_k))} = 0. \tag{6.21}
\end{aligned}$$

Here we have used $\oint_{|w_{j-1}|=1} \frac{dw_{j-1}}{w_{j-1}} (w_{j-1}^{-1} - w_{j-1})e^{-h_+^{j-1}(qw_{j-1})-h_+^{j-1}(q/w_{j-1})} = 0$.

- The case for $j = M+2$: ${}_{(1)}\langle B|\phi_{M+2}^*(z)\rangle$.

Using the bosonization (4.72), the relations (6.12), (6.13), (6.14), and the normal orderings in appendix C, we have LHS-RHS of (6.11) as following.

$$\begin{aligned}
& z^{\frac{M}{M-N}}\varphi^{(1)}(1/z) {}_{(1)}\langle B|\phi_{M+2}^*(z) - z^{-\frac{M}{M-N}}\varphi^{(1)}(z) {}_{(1)}\langle B|\phi_{M+2}^*(1/z) \\
&= q^{-\frac{M}{M-N}+1}(q - q^{-1})^M\varphi^{(1)}(z)\varphi^{(1)}(1/z)(z - z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
&\times \sum_{\epsilon=\pm} \epsilon \prod_{k=1}^{M+1} \int_{\tilde{C}_{M+2}} \frac{dw_k}{2\sqrt{-1}w_k} \frac{(w_1^{-1} - w_1)(1 + w_{M+1}^2)}{\prod_{k=0}^M D(w_k, w_{k+1})} \prod_{k=1}^M (1 - q/w_k w_{k+1}) \prod_{k=2}^M (1 - w_k^2) \\
&\times {}_{(1)}\langle B|e^{Q_{h_1^*-h_1}\dots-h_{M+1-c_1}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+1}(h_+^k(qw_k)+h_+^k(q/w_k))-c_+^1(q^{1+\epsilon}w_{M+1})-c_+^1(q^{1-\epsilon}/w_{M+1})}. \tag{6.22}
\end{aligned}$$

Here the integration contour \tilde{C}_{M+2} encircles $w_k = 0, qw_{k-1}^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq M+1$. Taking into account of symmetrization $\oint_{|w|=1} \frac{dw}{w} f(w) = \frac{1}{2} \oint_{|w|=1} (f(w) + f(1/w))$ and relation $(1 - q/w_1 w_2) - (w_1 \leftrightarrow w_1^{-1}) = (-q/w_2)(w_1^{-1} - w_1)$, we symmetrize the variables w_1, w_2, \dots, w_M iteratively. Then we have

$$\begin{aligned}
& (-q/2)^M q^{-\frac{M}{M-N}+1} (q - q^{-1})^M \varphi^{(1)}(z) \varphi^{(1)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
& \times \prod_{k=1}^{M+1} \int_{\tilde{C}_{M+2}} \frac{dw_k}{2\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2 (w_{M+1}^{-1} + w_{M+1})}{\prod_{k=0}^M D(w_k, w_{k+1})} \\
& \times {}_{(1)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{M+1} - c_1}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+1} (h_+^k(qw_k) + h_+^k(q/w_k))} \\
& \times \left(e^{-c_+^1(q^2 w_{M+1}) - c_+^1(1/w_{M+1})} - e^{-c_+^1(w_{M+1}) - c_+^1(q^2/w_{M+1})} \right) = 0. \tag{6.23}
\end{aligned}$$

Here we have used

$$\oint_{|w_{M+1}|=1} \frac{dw_{M+1}}{w_{M+1}} \frac{(w_{M+1} + w_{M+1}^{-1})}{D(w_M, w_{M+1})} \left(e^{-c_+^1(q^2 w_{M+1}) - c_+^1(1/w_{M+1})} - e^{-c_+^1(w_{M+1}) - c_+^1(q^2/w_{M+1})} \right) = 0.$$

- The case for $2 \leq j \leq N+1$: ${}_{(1)}\langle B | \phi_{M+1+j}^*(z)$.

Using the bosonization (4.73), the relations (6.12), (6.13), (6.14), and the normal orderings in appendix C, we have LHS–RHS of (6.11) as following.

$$\begin{aligned}
& z^{\frac{M}{M-N}} \varphi^{(1)}(1/z) {}_{(1)}\langle B | \phi_{M+j+1}^*(z) - z^{-\frac{M}{M-N}} \varphi^{(1)}(z) {}_{(1)}\langle B | \phi_{M+1+j}^*(1/z) \\
& = q^{-\frac{M}{M-N}+1} (q - q^{-1})^M \varphi^{(1)}(z) \varphi^{(1)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
& \times \sum_{\epsilon_1, \dots, \epsilon_j = \pm} \epsilon_j \prod_{k=1}^{M+j} \int_{\tilde{C}_{M+1+j}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{(w_1^{-1} - w_1)(1 + w_{M+1}^2)}{\prod_{k=0}^M D(w_k, w_{k+1})} \prod_{k=1}^M (1 - q/w_k w_{k+1}) \prod_{k=2}^M (1 - w_k^2) \\
& \times \prod_{k=1}^{j-1} \frac{\epsilon_k q^{\epsilon_k} (1 - qw_{M+k} w_{M+k+1})}{(1 - q^{\epsilon_k} w_{M+k} w_{M+k+1})(1 - q^{\epsilon_k} w_{M+k}/w_{M+k+1})} {}_{(1)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{M+j} - c_j}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z)} \\
& \times e^{-\sum_{k=1}^{M+j} (h_+^k(qw_k) + h_+^k(q/w_k)) + \sum_{k=1}^{j-1} (c_+^k(qw_{M+k+1}) + c_+^k(q/w_{M+k+1})) - \sum_{k=1}^j (c_+^k(q^{1+\epsilon_k} w_{M+k}) + c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \tag{6.24}
\end{aligned}$$

Here the integration contour \tilde{C}_{M+1+j} encircles $w_k = 0, qw_{k-1}^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq M+j$. Using relation $(1 - q/w_1 w_2) - (w_1 \leftrightarrow w_1^{-1}) = (-q/w_2)(w_1^{-1} - w_1)$ we symmetrize the variables w_1, w_2, \dots, w_M iteratively, we have

$$\begin{aligned}
& q^{-\frac{M}{M-N}} (-q/2)^M (q - q^{-1})^M \varphi^{(1)}(z) \varphi^{(1)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
& \times \sum_{\epsilon_1, \dots, \epsilon_j = \pm} \epsilon_j \prod_{k=1}^{M+j} \int_{\tilde{C}_{M+1+j}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2 (w_{M+1}^{-1} + w_{M+1})}{\prod_{k=0}^M D(w_k, w_{k+1})}
\end{aligned}$$

$$\begin{aligned}
& \times \prod_{k=1}^{j-1} \frac{\epsilon_k q^{\epsilon_k} (1 - q w_{M+k} w_{M+k+1})}{(1 - q^{\epsilon_k} w_{M+k} w_{M+k+1}) (1 - q^{\epsilon_k} w_{M+k} / w_{M+k+1})} {}_{(1)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{M+j} - c_j}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z)} \quad (6.25) \\
& \times e^{-\sum_{k=1}^{M+j} (h_+^k(qw_k) + h_+^k(q/w_k)) + \sum_{k=1}^{j-1} (c_+^k(qw_{M+k+1}) + c_+^k(q/w_{M+k+1})) - \sum_{k=1}^j (c_+^k(q^{1+\epsilon_k} w_{M+k}) + c_+^k(q^{1-\epsilon_k} / w_{M+k}))}.
\end{aligned}$$

Using the relation (6.16) for the variables $w_{M+1}, \dots, w_{M+j-1}$ iteratively, we have

$$\begin{aligned}
& q^{-\frac{M}{M-N}} (-q/2)^M (q - q^{-1})^M \varphi^{(1)}(z) \varphi^{(1)}(1/z) (z - z^{-1}) (-1)^{j-1} e^{\frac{\pi \sqrt{-1} M}{M-N}} \\
& \times \prod_{k=1}^{M+j} \int_{\tilde{C}_{M+1+j}} \frac{dw_k}{2\pi \sqrt{-1} w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2 (w_{M+1}^{-1} + w_{M+1})}{\prod_{k=0}^M D(w_k, w_{k+1})} \\
& \times {}_{(1)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{M+j} - c_j}} \prod_{k=1}^{j-1} \frac{(-1 + w_{M+k}^2) e^{-c_+^k(w_{M+k}) - c_+^k(q^2/w_{M+k})}}{(1 - w_{M+k} w_{M+k+1}/q) (1 - w_{M+k}/q w_{M+k+1})} \\
& \times e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+j} (h_+^k(qw_k) + h_+^k(q/w_k)) + \sum_{k=1}^{j-1} (c_+^k(qw_{M+k+1}) + c_+^k(q/w_{M+k+1}))} \\
& \times \left(e^{-c_+^j(q^2 w_{M+j}) - c_+^j(1/w_{M+j})} - e^{-c_+^j(q^2/w_{M+j}) - c_+^j(w_{M+j})} \right) = 0. \quad (6.26)
\end{aligned}$$

Here we have used

$$\oint_{|w|=1} \frac{dw}{w} e^{c_+^{j-1}(qw) + c_+^{j-1}(q/w)} \left(e^{-c_+^j(q^2 w) - c_+^j(1/w)} - e^{-c_+^j(w) - c_+^j(q^2/w)} \right) f(w) = 0,$$

where $f(w) = f(1/w)$.

Now we have shown the relation (6.11) for every $j = 1, 2, \dots, M + N + 2$.

Q.E.D.

6.2 Boundary condition 2

In this section we study (6.1) for the boundary condition ${}_{(2)}\langle B |$. Very explicitly we study

$$z^{\frac{M}{M-N}} \varphi^{(2)}(z^{-1}) {}_{(2)}\langle B | \phi_j^*(z) = z^{-\frac{M}{M-N}} \varphi^{(2)}(z) {}_{(2)}\langle B | \phi_j^*(z^{-1}) \quad (1 \leq j \leq L), \quad (6.27)$$

$$z^{\frac{M}{M-N}} (1 - rz) \varphi^{(2)}(z^{-1}) {}_{(2)}\langle B | \phi_j^*(z) = z^{-\frac{M}{M-N}} (1 - r/z) {}_{(2)}\langle B | \phi_j^*(z^{-1}) \quad (L + 1 \leq j \leq M + N + 2). \quad (6.28)$$

The structure of (6.27) is the same as those of (6.11) for the boundary condition ${}_{(1)}\langle B |$. In this section we focus our attention on the relation (6.28) that is new for the boundary condition ${}_{(2)}\langle B |$. We give proofs for following two conditions.

Condition 2.1 : $L \leq M + 1$,

Condition 2.2 : $M + 2 \leq L \leq M + N + 1$.

Proposition 6.5 *The actions of $h_+^i(w)$, $h_-^{*1}(w)$ and $c_-^j(w)$ on the boundary state ${}_{(2)}\langle B |$ are given as followings.*

$${}_{(2)}\langle B | e^{-h_-^{*1}(qw)} = g_i^{(2)}(w) {}_{(2)}\langle B | e^{-h_+^i(q/w)} \quad (i = 1, 2, \dots, M + N + 1), \quad (6.29)$$

$${}_{(2)}\langle B|e^{h^{*1}(qw)} = \varphi^{(2)}(w) {}_{(2)}\langle B|e^{h^{*1}(q/w)}, \quad (6.30)$$

$${}_{(2)}\langle B|e^{-c_-^j(qw)} = c_j^{(2)}(w) {}_{(2)}\langle B|e^{-c_+^j(q/w)} \quad (j = 1, 2, \dots, N+1). \quad (6.31)$$

Here $\varphi^{(2)}(w)$ are given in (2.22) and (2.23). We have set $c_j^{(2)}(w) = 1$ ($j = 1, 2, \dots, N+1$). We have set $g_i^{(2)}(w)$ ($i = 1, 2, \dots, M+N+1$) by

$$g_i^{(2)}(w) = \begin{cases} g_i^{(1)}(w) & (1 \leq i \neq L \leq M+N+1), \\ \frac{1}{(1-rq^{\alpha_L}w)}g_L^{(1)}(w) & (i = L), \end{cases} \quad (6.32)$$

where $g_i^{(1)}(w)$ is given by (6.15). The parameter α_L is given by followings.

Condition 2.1 : For $1 \leq L \leq M+1$ we have set

$$\alpha_L = -L. \quad (6.33)$$

Condition 2.2 : For $M+2 \leq L \leq M+N+2$ we have set

$$\alpha_L = -2M-2+L. \quad (6.34)$$

Proposition 6.6 *The following relation holds.*

$$\begin{aligned} & \sum_{\epsilon=\pm} \int_C \frac{dw_1}{w_1} \frac{\epsilon q^\epsilon (1-q^\alpha r w_1)(1-q w_1 w_2) e^{-c_+^j(q^{1+\epsilon} w_1) - c_+^j(q^{1-\epsilon}/w_1)}}{(1-q^\epsilon w_1 w_2)(1-q^\epsilon w_1/w_2)} \\ &= q(1-q^{\alpha+1} r w_2) \int_C \frac{dw_1}{w_1} \frac{(-1+w_1^2) e^{-c_+^j(w_1) - c_+^j(q^2/w_1)}}{(1-w_1 w_2/q)(1-w_1/q w_2)}, \end{aligned} \quad (6.35)$$

where the integration contour C encircles $w_1 = 0, q w_2^{\pm 1}$ not but $w_1 = q^{-1} w_2^{\pm 1}$.

The integral $\int_C \frac{dw_1}{w_1} \frac{(-1+w_1^2) e^{-c_+^j(w_1) - c_+^j(q^2/w_1)}}{(1-w_1 w_2/q)(1-w_1/q w_2)}$ in RHS is invariant under $w_2 \rightarrow w_2^{-1}$.

Proof for boundary condition 2.1 We show the main theorem for the boundary condition ${}_{(2)}\langle B|$ and $1 \leq L \leq M+1$. Here we show the relation (6.28).

• The case for $L+1 \leq j \leq M+1$: ${}_{(2)}\langle B|\phi_j^*(z)$.

Using the bosonization (4.71), the relations (6.29), (6.30), and the normal ordering in appendix C, we have LHS–RHS of (6.28) as following :

$$\begin{aligned} & z^{\frac{M}{M-N}} (1-rz) \varphi^{(2)}(1/z) {}_{(2)}\langle B|\phi_j^*(z) - z^{-\frac{M}{M-N}} (1-r/z) \varphi^{(2)}(z) {}_{(2)}\langle B|\phi_j^*(1/z) \\ &= q^{-\frac{M}{M-N}} (q - q^{-1})^{j-1} \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z - z^{-1}) e^{\frac{\pi \sqrt{-1}(j-1)}{M-N}} \\ &\times \prod_{k=1}^{j-1} \int_{\tilde{C}_j} \frac{dw_k}{2\pi \sqrt{-1} w_k} \frac{\prod_{k=1}^{j-2} (1 - q/w_k w_{k+1})}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} \frac{(w_1^{-1} - w_1)(1 - r q^{-1} w_1)}{(1 - r q^{-L} w_L)} \prod_{k=2}^{j-1} (1 - w_k^2) \\ &\times {}_{(2)}\langle B|e^{Q_{h_1^* - h_1 \dots - h_{j-1}}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{j-1} (h_+^k(q w_k) + h_+^k(q/w_k))}. \end{aligned} \quad (6.36)$$

Here the integration contour \tilde{C}_j encircles $w_k = 0, q w_{k-1}^{\pm 1}$ not but $w_k = q^{-1} w_{k-1}^{\pm 1}$ for $1 \leq k \leq j-1$.

Taking into account of symmetrization $\oint_{|w|=1} \frac{dw}{w} f(w) = \frac{1}{2} \oint_{|w|=1} \frac{dw}{w} (f(w) + f(1/w))$ and relation (1 -

$q/w_1 w_2)(1 - rq^{-\alpha} w_1) + (w_1 \leftrightarrow w_1^{-1}) = (w_1^{-1} - w_1)(-q/w_2)(1 - rq^{-\alpha-1} w_2)$, we symmetrize the variables w_1, w_2, \dots, w_{L-1} , iteratively. Using relation $(1 - q/w_L w_{L+1}) - (w_L \leftrightarrow w_L^{-1}) = (-q/w_{L+1})(w_L^{-1} - w_L)$, we symmetrize the variables $w_L, w_{L+1}, \dots, w_{j-1}$, iteratively, we have

$$\begin{aligned} & q^{-\frac{M}{M-N}} (q - q^{-1})^{j-1} (-q/2)^{j-1} \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}(j-1)}{M-N}} \\ & \times \prod_{k=1}^{j-1} \int_{\tilde{C}_j} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{j-2} (w_k^{-1} - w_k)^2}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} (w_{j-1}^{-1} - w_{j-1}) \\ & \times {}_{(2)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{j-1}}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{j-1} (h_+^k(qw_k) + h_+^k(q/w_k))} = 0. \end{aligned} \quad (6.37)$$

Here we have used $\oint_{|w|=1} \frac{dw}{w} (w^{-1} - w) f(w) = 0$, where $f(w) = f(1/w)$.

- The case for $j = L + 1 = M + 2$: ${}_{(2)}\langle B | \phi_{M+2}^*(z)$.

Using the bosonization (4.72), relations (6.29), (6.30), (6.31), and the normal orderings in appendix C, we have LHS–RHS of (6.28) as following :

$$\begin{aligned} & z^{\frac{M}{M-N}} \varphi^{(2)}(1/z) (1 - rz) {}_{(2)}\langle B | \phi_{M+2}^*(z) - z^{-\frac{M}{M-N}} \varphi^{(2)}(z) (1 - r/z) {}_{(2)}\langle B | \phi_{M+2}^*(1/z) \\ & = q^{-\frac{M}{M-N}+1} (q - q^{-1})^M \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\ & \times \sum_{\epsilon=\pm} \epsilon \prod_{k=1}^{M+1} \int_{\tilde{C}_{M+2}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (1 - q/w_k w_{k+1})}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(w_1^{-1} - w_1)(1 - rq^{-1}w_1)(1 + w_{M+1}^2)}{(1 - rq^{-L}w_L)} \\ & \times \prod_{k=2}^M (1 - w_k^2) \times {}_{(2)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{L-c_1}}} \\ & \times e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^L (h_+^k(qw_k) + h_+^k(q/w_k)) - c_+^1(q^{1+\epsilon}w_{M+1}) - c_+^1(q^{1-\epsilon}/w_{M+1})}. \end{aligned} \quad (6.38)$$

Here the integration contour \tilde{C}_{M+2} encircles $w_k = 0, qw_{k-1}^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq M + 1$. Using relation $(1 - rq^\alpha w_1)(1 - q/w_1 w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})(1 - rq^{\alpha-1} w_2)$, we symmetrize the variables $w_1, w_2, \dots, w_L = w_{M+1}$, iteratively, we have

$$\begin{aligned} & q^{-\frac{M}{M-N}+1} (q - q^{-1})^M (-q/2)^M \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\ & \times \int_{\tilde{C}_{M+2}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} (w_{M+1}^{-1} + w_{M+1}) \times {}_{(2)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{L-c_1}}} \\ & \times e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^L (h_+^k(qw_k) + h_+^k(q/w_k))} \left(e^{-c_+^1(q^2 w_{M+1}) - c_+^1(1/w_{M+1})} - e^{-c_+^1(w_{M+1}) - c_+^1(q^2/w_{M+1})} \right) = 0. \end{aligned} \quad (6.39)$$

Here we have used $\oint_{|w|=1} \frac{dw}{w} (e^{-c_+^1(q^2 w) - c_+^1(1/w)} - e^{-c_+^1(w) - c_+^1(q^2/w)}) f(w) = 0$, where $f(w) = f(1/w)$.

- The case for $M + 3 \leq j \leq M + N + 2$: ${}_{(2)}\langle B | \phi_j^*(z)$.

Using the bosonization (4.73), the relations (6.29), (6.30), (6.31), and the normal orderings in appendix

C, we have LHS–RHS of (6.11) for $1 \leq i \leq N+1$.

$$\begin{aligned}
& z^{\frac{M}{M-N}} (1-rz) \varphi^{(2)}(1/z) {}_{(2)}\langle B | \phi_{M+1+i}^*(z) - z^{-\frac{M}{M-N}} (1-r/z) \varphi^{(2)}(z) {}_{(2)}\langle B | \phi_{M+1+i}^*(1/z) \\
&= q^{-\frac{M}{M-N}+1} (q-q^{-1})^M \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z-z^{-1}) e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
&\times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (1-q/w_k w_{k+1})}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(w_1^{-1} - w_1)(1-rq^{-1}w_1)(1+w_{M+1}^2)}{(1-rq^{-L}w_L)} \\
&\times \prod_{k=2}^M (1-w_k^2) \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})} \\
&\times {}_{(2)}\langle B | e^{Q_{h_1^*-h_1 \dots -h_{M+i-c_i}}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))} \\
&\times e^{\sum_{k=1}^{i-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))- \sum_{k=1}^i(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \tag{6.40}
\end{aligned}$$

Here the integration contour \tilde{C}_{M+1+i} encircles $w_k = 0, qw_{k-1}^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq M+i$. Using relation $(1-rq^\alpha w_1)(1-q/w_1 w_2) - (w_1 \leftrightarrow w_1^{-1}) = (1-rq^{\alpha-1}w_2)(w_1 - w_1^{-1})$, we symmetrize the variables w_1, w_2, \dots, w_{L-1} , iteratively. Using relation $(1-q/w_L w_{L+1}) - (w_L \leftrightarrow w_L^{-1}) = (-q/w_{L+1})(w_L^{-1} - w_L)$ we symmetrize the variables w_L, \dots, w_M , iteratively. Then we have

$$\begin{aligned}
& q^{-\frac{M}{M-N}+1} (q-q^{-1})^M (-q/2)^M \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z-z^{-1}) e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
&\times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} (w_{M+1}^{-1} + w_{M+1}) \\
&\times \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1-qw_{M+k}w_{M+k+1})}{(1-q^{\epsilon_k}w_{M+k}w_{M+k+1})(1-q^{\epsilon_k}w_{M+k}/w_{M+k+1})} {}_{(2)}\langle B | e^{Q_{h_1^*-h_1 \dots -h_{M+i-c_i}}} e^{h_+^{*1}(qz)+h_+^{*1}(q/z)} \\
&\times e^{-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{i-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))- \sum_{k=1}^i(c_+^k(q^{1+\epsilon_k}w_{M+k})+c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \tag{6.41}
\end{aligned}$$

Using relation (6.16) for variables w_{M+1}, \dots, w_{i-1} , iteratively, we have

$$\begin{aligned}
& q^{-\frac{M}{M-N}+1} (q-q^{-1})^M (-q/2)^M \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z-z^{-1}) e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
&\times \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} (w_{M+1}^{-1} + w_{M+1}) \times {}_{(2)}\langle B | e^{Q_{h_1^*-h_1 \dots -h_{M+i-c_i}}} \\
&\times \prod_{k=1}^{i-1} \frac{(-1+w_{M+k}^2) e^{-c_+^k(w_{M+k})-c_+^k(q^2/w_{M+k})}}{(1-w_{M+k}w_{M+k+1}/q)(1-w_{M+k}/qw_{M+k+1})} \\
&\times e^{h_+^{*1}(qz)+h_+^{*1}(q/z)-\sum_{k=1}^{M+i}(h_+^k(qw_k)+h_+^k(q/w_k))+\sum_{k=1}^{i-1}(c_+^k(qw_{M+k+1})+c_+^k(q/w_{M+k+1}))} \\
&\times \left(e^{-c_+^i(q^2w_{M+i})-c_+^i(1/w_{M+i})} - e^{-c_+^i(w_{M+i})-c_+^i(q^2/w_{M+i})} \right) = 0. \tag{6.42}
\end{aligned}$$

Here we have used

$$\oint_{|w|=1} \frac{dw}{w} e^{c_+^{i-1}(qw) + c_+^{i-1}(q/w)} \left(e^{-c_+^i(q^2w) - c_+^i(1/w)} - e^{-c_+^i(w) - c_+^i(q^2/w)} \right) f(w) = 0,$$

where $f(w) = f(1/w)$.

Now we have shown (6.28) for every $j = L + 1, \dots, M + N + 2$.

Q.E.D.

Proof for boundary condition 2.2 We show the main theorem for the boundary condition ${}_{(2)}\langle B|$ and $M + 2 \leq L \leq M + N + 1$. Here we show (6.28) that is new for the boundary condition ${}_{(2)}\langle B|$. We use the integral relations (6.16) and (6.35).

• The case for $L + 1 \leq j \leq M + N + 2 : {}_{(2)}\langle B|\phi_j^*(z)$.

Using the bosonization (4.73), the relations (6.29), (6.30), (6.31), and the normal orderings in appendix C, we have LHS–RHS of (6.11) for $1 \leq i \leq N + 1$.

$$\begin{aligned} & z^{\frac{M}{M-N}} (1 - rz) \varphi^{(2)}(1/z) {}_{(2)}\langle B|\phi_{M+1+i}^*(z) - z^{-\frac{M}{M-N}} (1 - r/z) \varphi^{(2)}(z) {}_{(2)}\langle B|\phi_{M+1+i}^*(1/z) \\ &= q^{-\frac{M}{M-N}+1} (q - q^{-1})^M \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}M}{M-N}} \\ &\times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (1 - q/w_k w_{k+1})}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(w_1^{-1} - w_1)(1 - rq^{-1}w_1)(1 + w_{M+1}^2)}{(1 - rq^{-2M-2+L}w_L)} \\ &\times \prod_{k=2}^M (1 - w_k^2) \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1 - qw_{M+k} w_{M+k+1})}{(1 - q^{\epsilon_k} w_{M+k} w_{M+k+1})(1 - q^{\epsilon_k} w_{M+k}/w_{M+k+1})} \\ &\times {}_{(2)}\langle B| e^{Q_{h_1^* - h_1 \dots - h_{M+i} - c_i}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k))} \\ &\times e^{\sum_{k=1}^{i-1} (c_+^k(qw_{M+k+1}) + c_+^k(q/w_{M+k+1})) - \sum_{k=1}^i (c_+^k(q^{1+\epsilon_k} w_{M+k}) + c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \end{aligned} \quad (6.43)$$

Here the integration contour \tilde{C}_{M+1+i} encircles $w_k = 0, qw_{k-1}^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq M + i$. Using relation $(1 - q/w_1 w_2)(1 - rq^{-\alpha} w_1) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})(1 - rq^{-\alpha-1} w_2)$, we symmetrize the variables w_1, w_2, \dots, w_M iteratively. We have

$$\begin{aligned} & q^{-\frac{M}{M-N}+1} (q - q^{-1})^M (-q/2)^M \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}M}{M-N}} \\ &\times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(w_{M+1}^{-1} + w_{M+1})(1 - rq^{-M-1}w_{M+1})}{(1 - rq^{-2M-2+L}w_L)} \\ &\times \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1 - qw_{M+k} w_{M+k+1})}{(1 - q^{\epsilon_k} w_{M+k} w_{M+k+1})(1 - q^{\epsilon_k} w_{M+k}/w_{M+k+1})} \\ &\times {}_{(2)}\langle B| e^{Q_{h_1^* - h_1 \dots - h_{M+i} - c_i}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k))} \\ &\times e^{\sum_{k=1}^{i-1} (c_+^k(qw_{M+k+1}) + c_+^k(q/w_{M+k+1})) - \sum_{k=1}^i (c_+^k(q^{1+\epsilon_k} w_{M+k}) + c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \end{aligned} \quad (6.44)$$

We use the relation (6.35) for the variables w_{M+1}, \dots, w_{L-1} iteratively. We use the relation (6.16) for the variables w_L, \dots, w_{M+i-1} iteratively. Then we have

$$\begin{aligned}
& q^{-\frac{M}{M-N}+1}(q-q^{-1})^M(-q/2)^M q^{L-M-1} \varphi^{(2)}(z) \varphi^{(2)}(1/z) (z-z^{-1}) e^{\frac{\pi\sqrt{-1}M}{M-N}} \\
& \times \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} (w_{M+1}^{-1} + w_{M+1}) \\
& \times {}_{(2)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{M+i} - c_i}} \prod_{k=1}^{i-1} \frac{(-1 + w_{M+k}^2) e^{-c_+^k(w_{M+k}) - c_+^k(q^2/w_{M+k})}}{(2 - w_{M+k}/q w_{M+k+1})(1 - w_{M+k} w_{M+k+1}/q)} \\
& \times e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k)) + \sum_{k=1}^{i-1} (c_+^k(qw_{M+k+1}) + c_+^k(q/w_{M+k+1}))} \\
& \times \left(e^{-c_+^i(q^2 w_{M+i}) - c_+^i(1/w_{M+i})} - e^{-c_+^i(w_{M+i}) - c_+^i(q^2/w_{M+i})} \right) = 0. \tag{6.45}
\end{aligned}$$

Here we have used

$$\oint_{|w|=1} \frac{dw}{w} e^{c_+^{i-1}(qw) + c_+^{i-1}(q/w)} \left(e^{-c_+^i(q^2 w) - c_+^i(1/w)} - e^{-c_+^i(w) - c_+^i(q^2/w)} \right) f(w) = 0,$$

where $f(w) = f(1/w)$.

Q.E.D.

Now we have shown (6.28) for every $j = L+1, \dots, M+N+2$.

6.3 Boundary condition 3

In this section we study (6.1) for the boundary condition ${}_{(3)}\langle B |$. Very explicitly we study

$$z^{\frac{M}{M-N}} \varphi^{(3)}(z^{-1}) {}_{(3)}\langle B | \phi_j^*(z) = z^{-\frac{M}{M-N}} \varphi^{(3)}(z) {}_{(3)}\langle B | \phi_j^*(z^{-1}) \quad (1 \leq j \leq L), \tag{6.46}$$

$$z^{\frac{M}{M-N}} (1-rz) \varphi^{(3)}(z^{-1}) {}_{(3)}\langle B | \phi_j^*(z) = z^{-\frac{M}{M-N}} (1-r/z) \varphi^{(3)}(z) {}_{(3)}\langle B | \phi_j^*(z^{-1}) \quad (L+1 \leq j \leq L+K), \tag{6.47}$$

$$z^{\frac{M}{M-N}+1} \varphi^{(3)}(z^{-1}) {}_{(3)}\langle B | \phi_j^*(z) = z^{-\frac{M}{M-N}-1} \varphi^{(3)}(z) {}_{(3)}\langle B | \phi_j^*(z^{-1}) \quad (L+K+1 \leq j \leq M+N+2). \tag{6.48}$$

The structures of (6.46) and (6.47) are the same as those of (6.27) and (6.28) for the boundary condition ${}_{(2)}\langle B |$. In this section we focus our attention on the relation (6.48) that is new for the boundary condition ${}_{(3)}\langle B |$. We give proofs for following three conditions.

Condition 3.1 : $L+K \leq M+1$,

Condition 3.2 : $L \leq M+1 \leq L+K-1$,

Condition 3.3 : $M+2 \leq L$.

Proposition 6.7 *The actions of $h_+^i(w)$, $h_-^{*1}(w)$ and $c_-^j(w)$ on the boundary state ${}_{(3)}\langle B |$ are given as followings.*

$${}_{(3)}\langle B | e^{-h_-^{*1}(qw)} = g_i^{(3)}(w) {}_{(3)}\langle B | e^{-h_+^i(q/w)} \quad (i = 1, 2, \dots, M+N+1), \tag{6.49}$$

$${}_{(3)}\langle B|e^{h^{*-1}(qw)} = \varphi^{(3)}(w) {}_{(3)}\langle B|e^{h^{*1}(q/w)}, \quad (6.50)$$

$${}_{(3)}\langle B|e^{-c_-^j(qw)} = c_j^{(3)}(w) {}_{(3)}\langle B|e^{-c_+^j(q/w)} \quad (j = 1, 2, \dots, N+1). \quad (6.51)$$

Here $\varphi^{(3)}(w)$ are given in (2.25), (2.26) and (2.27). We have set $c_j^{(3)}(w) = 1$ ($j = 1, 2, \dots, N+1$). We have set $g_i^{(3)}(w)$ ($i = 1, 2, \dots, M+N+1$) by

$$g_i^{(3)}(w) = \begin{cases} g_i^{(1)}(w) & (1 \leq i \neq L, L+K \leq M+N+1), \\ \frac{1}{(1-rq^{\alpha_L}w)} g_L^{(1)}(w) & (i = L), \\ \frac{1}{(1-q^{\alpha_{L+K}}w/r)} g_{L+K}^{(1)}(w) & (i = L+K), \end{cases} \quad (6.52)$$

where $g_i^{(1)}(w)$ is given by (6.15). The parameters (α_L, α_{L+K}) are given by followings.

Condition 3.1 : For $L+K \leq M+1$ we have set

$$(\alpha_L, \alpha_{L+K}) = (-L, L-K). \quad (6.53)$$

Condition 3.2 : For $L \leq M+1 \leq L+K-1$ we have set

$$(\alpha_L, \alpha_{L+K}) = (-L, 3L+K-2M-2), \quad (6.54)$$

Condition 3.3 : For $M+2 \leq L$ we have set

$$(\alpha_L, \alpha_{L+K}) = (L-2M-2, 2M+K-L+2). \quad (6.55)$$

Proposition 6.8 We have the following two relations.

$$\begin{aligned} & \sum_{\epsilon=\pm} \int_C \frac{dw_1}{w_1} \frac{\epsilon q^\epsilon w_1 (1-qw_1w_2) e^{-c_+^j(q^{1+\epsilon}w_1)-c_+^j(q^{1-\epsilon}/w_1)}}{(1-rq^\alpha w_1)(1-q^\epsilon w_1w_2)(1-q^\epsilon w_1/w_2)} \\ &= -rq^{\alpha-1}(1-q^{-\alpha+1}w_2/r) \int_C \frac{dw_1}{w_1} \frac{(-1+w_1^2) e^{-c_+^j(w_1)-c_+^j(q^2/w_1)}}{(1-w_1w_2/q)(1-w_1/qw_2)(1-rq^\alpha w_1)(1-rq^\alpha/w_1)}, \\ & \sum_{\epsilon=\pm} \int_C \frac{dw_1}{w_1} \frac{\epsilon q^\epsilon w_1 (1-qw_1w_2) e^{-c_+^j(q^{1+\epsilon}w_1)-c_+^j(q^{1-\epsilon}/w_1)}}{(1-q^\epsilon w_1w_2)(1-q^\epsilon w_1/w_2)} \\ &= w_2 \int_C \frac{dw_1}{w_1} \frac{(-1+w_1^2) e^{-c_+^j(w_1)-c_+^j(q^2/w_1)}}{(1-w_1w_2/q)(1-w_1/qw_2)}. \end{aligned} \quad (6.56)$$

Here the integration contour C encircles $w_1 = 0, qw_2^{\pm 1}$ not but $w_1 = q^{-1}w_2^{\pm 1}$. Here the integrals $\int_C \frac{dw_1}{w_1} \frac{(-1+w_1^2) e^{-c_+^j(w_1)-c_+^j(q^2/w_1)}}{(1-w_1w_2/q)(1-w_1/qw_2)(1-rq^\alpha w_1)(1-rq^\alpha/w_1)}$ and $\int_C \frac{dw_1}{w_1} \frac{(-1+w_1^2) e^{-c_+^j(w_1)-c_+^j(q^2/w_1)}}{(1-w_1w_2/q)(1-w_1/qw_2)}$ in RHS are invariant under $w_2 \rightarrow w_2^{-1}$.

Proof for boundary condition 3.1 We show the main theorem for the boundary condition ${}_{(3)}\langle B|$ and $L+K \leq M+1$. Here we show the relation (6.48).

- The case for $1 \leq L+K+1 \leq j \leq M+1$: ${}_{(3)}\langle B|\phi_j^*(z)$.

Using the bosonization (4.71), the relations (6.49), (6.50), (6.51), and the normal orderings in appendix

C, we have LHS–RHS of (6.48) as following :

$$\begin{aligned}
& z^{\frac{M}{M-N}+1} \varphi^{(3)}(1/z) {}_{(3)}\langle B | \phi_{M+1+i}^*(z) - z^{-\frac{M}{M-N}-1} \varphi^{(3)}(z) {}_{(3)}\langle B | \phi_{M+1+i}(1/z) \\
&= q^{-\frac{M}{M-N}-1} \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}(j-1)} \\
&\times \prod_{k=1}^{j-1} \int_{\tilde{C}_j} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{j-2} (1 - q/w_k w_{k+1})}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} \frac{\prod_{k=1}^{j-1} (1 - w_k^2)}{(1 - q^{-L} r w_L) (q^{L-K} w_{L+K}/r)} \\
&\times {}_{(3)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{j-1}}} e^{h_+^*(qz) + h_+^*(q/z) - \sum_{k=1}^{j-1} (h_+^k(qw_k) + h_+^k(q/w_k))}. \tag{6.58}
\end{aligned}$$

Here the integration contour \tilde{C}_j encircles $w_k = 0, qw_{k-1}^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq j$. Taking into account of symmetrization $\oint \frac{dw}{w} f(w) = \frac{1}{2} \oint (f(w) + f(1/w))$ and relation $w_1(1 - q/w_1 w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})$, we symmetrize the variables w_1, w_2, \dots, w_{L-1} iteratively. Using relation $\frac{w_L(1 - q/w_L w_{L+1})}{(1 - rq^{-L} w_L)} - (w_L \leftrightarrow w_L^{-1}) = \frac{(w_L - w_L^{-1})(1 - rq^{-L+1}/w_{L+1})}{(1 - rq^{-L} w_L)(1 - rq^{-L}/w_L)}$, we symmetrize the variable w_L . Using relation $(1 - rq^\alpha/w_{L+1})w_{L+1}(1 - q/w_{L+1} w_{L+2}) - (w_{L+1} \leftrightarrow w_{L+1}^{-1}) = (w_{L+1} - w_{L+1}^{-1})(1 - rq^{\alpha+1}/w_{L+2})$, we symmetrize the variable $w_{L+1}, \dots, w_{L+K-1}$ iteratively. Using relation $(1 - q/w_{L+K+1} w_{L+K+2}) - (w_{L+K+1} \leftrightarrow w_{L+K+1}^{-1}) = (-q/w_{L+K+2})(w_{L+K+1}^{-1} - w_{L+K+1})$, we symmetrize the variables $w_{L+K+1}, \dots, w_{j-2}$ iteratively. Then we have

$$\begin{aligned}
& q^{-\frac{M}{M-N}+j-2L-3} (-1/2)^{j-2} \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}(j-1)} \\
&\times \prod_{k=1}^{j-1} \int_{\tilde{C}_j} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^{j-2} (w_k^{-1} - w_k)^2}{\prod_{k=0}^{j-2} D(w_k, w_{k+1})} \frac{(w_{j-1}^{-1} - w_{j-1})}{(1 - q^{-L} r w_L) (q^{-L} r/w_L)} \\
&\times {}_{(3)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{j-1}}} e^{h_+^*(qz) + h_+^*(q/z) - \sum_{k=1}^{j-1} (h_+^k(qw_k) + h_+^k(q/w_k))} = 0. \tag{6.59}
\end{aligned}$$

Here we have used $\oint_{|w|=1} \frac{dw}{w} (w - w^{-1}) f(w) = 0$, where $f(w) = f(1/w)$.

• The case for $1 \leq i \leq N+1$: ${}_{(3)}\langle B | \phi_{M+1+i}^*(z)$.

Using the bosonization (4.73), the relations (6.49), (6.50), (6.51), and normal orderings in appendix C, we have LHS–RHS of (6.48) as following :

$$\begin{aligned}
& z^{\frac{M}{M-N}+1} \varphi^{(3)}(1/z) {}_{(3)}\langle B | \phi_{M+1+i}^*(z) - z^{-\frac{M}{M-N}-1} \varphi^{(3)}(z) {}_{(3)}\langle B | \phi_{M+1+i}(1/z) \\
&= q^{-\frac{M}{M-N}-1} (q - q^{-1})^M \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
&\times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (1 - q/w_k w_{k+1})}{\prod_{k=0}^M D(w_k, w_{k+1})} \prod_{k=1}^M (1 - w_k^2) \\
&\times \frac{(1 + w_{M+1}^2)}{(1 - rq^{-L} w_L)(1 - q^{L-K} w_{L+K}/r)} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1 - q w_{M+k} w_{M+k+1})}{(1 - q^{\epsilon_k} w_{M+k} w_{M+k+1})(1 - q^{\epsilon_k} w_{M+k}/w_{M+k+1})} \\
&\times {}_{(3)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{M+i} - c_i}} e^{h_+^*(qz) + h_+^*(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k))}
\end{aligned}$$

$$\times e^{\sum_{k=1}^{i-1} (c_+^k(qw_{M+k}) + c_+^k(q/w_{M+k})) - \sum_{k=1}^i (c_+^k(q^{1+\epsilon_k}w_{M+k}) + c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \quad (6.60)$$

Here the integration contour \tilde{C}_{M+1+i} encircles $w_k = 0, qw_{k-1}^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq M+i$. Using relation $w_1(1 - q/w_1w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})$, we symmetrize the variables w_1, w_2, \dots, w_{L-1} iteratively. Using relation $\frac{w_L(1-q/w_Lw_{L+1})}{(1-rq^{-L}w_L)} - (w_L \leftrightarrow w_L^{-1}) = \frac{(w_L - w_L^{-1})(1-rq^{-L+1}/w_{L+1})}{(1-rq^{-L}w_L)(1-rq^{-L}/w_L)}$, we symmetrize the variable w_L . Using relation $(1 - rq^s/w_{L+1})w_{L+1}(1 - q/w_{L+1}w_{L+2}) - (w_{L+1} \leftrightarrow w_{L+1}^{-1}) = (w_{L+1} - w_{L+1}^{-1})(1 - rq^{s+1}/w_{L+2})$, we symmetrize the variable $w_{L+1}, \dots, w_{L+K-1}$ iteratively. Using relation $(1 - q/w_{L+K+1}w_{L+K+2}) - (w_{L+K+1} \leftrightarrow w_{L+K+1}^{-1}) = (-q/w_{L+K+2})(w_{L+K+1}^{-1} - w_{L+K+1})$, we symmetrize the variables $w_{L+K+1}, \dots, w_{M-1}$ iteratively. Then we have

$$\begin{aligned} & q^{-\frac{M}{M-N} - M - 2L - 2} (q - q^{-1})^M (-1/2)^M \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\ & \times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(w_{M+1}^{-1} + w_{M+1})}{(1 - rq^{-L}w_L)(1 - rq^{-L}/w_L)} \\ & \times \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1 - qw_{M+k}w_{M+k+1})}{(1 - q^{\epsilon_k}w_{M+k}w_{M+k+1})(1 - q^{\epsilon_k}w_{M+k}/w_{M+k+1})} \\ & \times {}_{(3)}\langle B | e^{Q_{h_1^* - h_1, \dots, h_{M+i} - c_i}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k))} \\ & \times e^{\sum_{k=1}^{i-1} (c_+^k(qw_{M+k}) + c_+^k(q/w_{M+k})) - \sum_{k=1}^i (c_+^k(q^{1+\epsilon_k}w_{M+k}) + c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \end{aligned} \quad (6.61)$$

Using the relation (6.16) for the variables $w_{M+1}, \dots, w_{M+i-1}$, iteratively, we have

$$\begin{aligned} & q^{-\frac{M}{M-N} + M - 2L - 2} (q - q^{-1})^M (-1/2)^{M-1} \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\ & \times \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(w_{M+1}^{-1} + w_{M+1})}{(1 - rq^{-L}w_L)(1 - q^{L-K}w_{L+K}/r)} \\ & \times {}_{(3)}\langle B | e^{Q_{h_1^* - h_1, \dots, h_{M+i} - c_i}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k)) + \sum_{k=1}^{i-1} (c_+^k(qw_{M+k}) + c_+^k(q/w_{M+k}))} \\ & \times \frac{\prod_{k=1}^{i-1} (-1 + w_{M+k}^2) e^{-c_+^k(w_{M+k}) - c_+^k(q^2/w_{M+k})}}{(1 - q^{\epsilon_k}w_{M+k}w_{M+k+1})(1 - q^{\epsilon_k}w_{M+k}/w_{M+k+1})} \\ & \times \left(e^{-c_+^i(q^2w_{M+i}) - c_+^i(q/w_{M+i}) - e^{-c_+^i(w_{M+i}) - c_+^i(q^2/w_{M+i})}} \right) = 0. \end{aligned} \quad (6.62)$$

Here we have used $\oint_{|w|=1} \frac{dw}{w} e^{c_+^{i-1}(qw) + c_+^{i-1}(1/w)} (e^{-c_+^i(q^2w) - c_+^i(1/w)} - e^{-c_+^i(w) - c_+^i(q^2/w)}) f(w) = 0$, where $f(w) = f(1/w)$.

Q.E.D.

Proof for boundary condition 3.2 We show the main theorem for the boundary condition ${}_{(3)}\langle B |$ and $L \leq M+1 \leq L+K-1$. Here we show the relation (6.48).

- The case for $L+K-M \leq i \leq N+1$: ${}_{(3)}\langle B | \phi_{M+1+i}^*(z)$.

Using the bosonization (4.73), the relations (6.49), (6.50), (6.51), and normal orderings in appendix C, we have LHS–RHS of (6.48) as following :

$$\begin{aligned}
& z^{\frac{M}{M-N}+1} \varphi^{(3)}(1/z) {}_{(3)}\langle B | \phi_{M+1+i}^*(z) - z^{\frac{M}{M-N}-1} \varphi^{(3)}(z) {}_{(3)}\langle B | \phi_{M+1+i}^*(1/z) \\
&= q^{-\frac{M}{M-N}-1} (q - q^{-1})^M \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
&\times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (1 - q/w_k w_{k+1})}{\prod_{k=0}^M D(w_k, w_{k+1})} \prod_{k=1}^M (1 - w_k^2) \\
&\times \frac{(1 + w_{M+1}^2)}{(1 - rq^{-L}w_L)(1 - q^{3L+K-2M+K-2}w_{L+K}/r)} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1 - qw_{M+k}w_{M+k+1})}{(1 - q^{\epsilon_k}w_{M+k}w_{M+k+1})(1 - q^{\epsilon_k}w_{M+k}/w_{M+k+1})} \\
&\times {}_{(3)}\langle B | e^{Q_{h_1^*-h_1 \dots - h_{M+i-c_i}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k))}} \\
&\times e^{\sum_{k=1}^{i-1} (c_+^k(qw_{M+k}) + c_+^k(q/w_{M+k})) - \sum_{k=1}^i (c_+^k(q^{1+\epsilon_k}w_{M+k}) + c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \tag{6.63}
\end{aligned}$$

Here the integration contour \tilde{C}_{M+1+i} encircles $w_k = 0, qw_{k-1}^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq M+i$. Using relation $w_1(1 - q/w_1 w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})$, we symmetrize the variables w_1, w_2, \dots, w_{L-1} , iteratively. Using relation $\frac{w_L(1 - q/w_L w_{L+1})}{(1 - rq^{-L}w_L)} - (w_L \leftrightarrow w_L^{-1}) = \frac{(w_L^{-1} - w_L)(1 - rq^{-L+1}/w_{L+1})}{(1 - rq^{-L}w_L)(1 - rq^{-L}/w_L)}$, we symmetrize the variables w_L . Using relation $w_{L+1}(1 - rq^{-\alpha}/w_{L+1})(1 - q/w_{L+1}w_{L+2}) - (w_{L+1} \leftrightarrow w_{L+1}^{-1}) = (w_{L+1} - w_{L+1}^{-1})(1 - rq^{-\alpha+1}/w_{L+2})$, we symmetrize the variables w_{L+1}, \dots, w_M , iteratively. Then we have

$$\begin{aligned}
& q^{-\frac{M}{M-N}-1} (-1/2)^M (q - q^{-1})^M \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
&\times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(1 - rq^{M-2L+1}/w_{M+1})}{(1 - rq^{-L}w_L)(1 - rq^{-L}/w_L)} \\
&\times \frac{(1 + w_{M+1}^2)}{(1 - q^{3L+K-2M-2}w_{L+K}/r)} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1 - qw_{M+k}w_{M+k+1})}{(1 - q^{\epsilon_k}w_{M+k}w_{M+k+1})(1 - q^{\epsilon_k}w_{M+k}/w_{M+k+1})} \\
&\times {}_{(3)}\langle B | e^{Q_{h_1^*-h_1 \dots - h_{M+i-c_i}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k))}} \\
&\times e^{\sum_{k=1}^{i-1} (c_+^k(qw_{M+k}) + c_+^k(q/w_{M+k})) - \sum_{k=1}^i (c_+^k(q^{1+\epsilon_k}w_{M+k}) + c_+^k(q^{1-\epsilon_k}/w_{M+k}))}. \tag{6.64}
\end{aligned}$$

Using the relation (6.35), we symmetrize the variables $w_{M+1}, \dots, w_{L+K-1}$ iteratively. Using the relation (6.16), we symmetrize the variables w_{L+K}, \dots, w_{M+i} iteratively. Then we have

$$\begin{aligned}
& (-r)q^{-\frac{M}{M-N}+K-L-1} (-1/2)^M (q - q^{-1})^M \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
&\times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(w_{M+1}^{-1} + w_{M+1})}{(1 - rq^{-L}w_L)(1 - rq^{-L}/w_L)} \\
&\times {}_{(3)}\langle B | e^{Q_{h_1^*-h_1 \dots - h_{M+i-c_i}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k)) + \sum_{k=1}^{i-1} (c_+^k(qw_{M+k}) + c_+^k(q/w_{M+k}))}}
\end{aligned}$$

$$\begin{aligned}
& \times \prod_{k=1}^{i-1} \frac{(-1 - qw_{M+k}^2) e^{-c_+^k(w_{M+k}) - c_+^k(q^2/w_{M+k})}}{(1 - w_{M+k} w_{M+k+1}/q)(1 - w_{M+k}/qw_{M+k+1})} \\
& \times \left(e^{-c_+^i(q^2 w_{M+i}) - c_+^i(1/w_{M+i})} - e^{-c_+^i(w_{M+i}) - c_+^i(q^2/w_{M+i})} \right) = 0.
\end{aligned} \tag{6.65}$$

Here we have used

$$\oint_{|w|=1} \frac{dw}{w} e^{c_+^{i-1}(qw) + c_+^{i-1}(q/w)} \left(e^{-c_+^i(q^2 w) - c_+^i(1/w)} - e^{-c_+^i(w) - c_+^i(q^2/w)} \right) f(w) = 0,$$

where $f(w) = f(1/w)$.

Q.E.D.

Proof for boundary condition 3.3 We show the main theorem for the boundary condition ${}_{(3)}\langle B|$ and $1 \leq M+1 \leq L-1$. Here we show the relation (6.48).

• The case for $L+K-M \leq i \leq N+1$: ${}_{(3)}\langle B|\phi_{M+1+i}^*(z)$.

Using the bosonization (4.73) and the relations (6.49), (6.50), (6.51), and normal orderings in appendix C, we have LHS–RHS of (6.48) as following :

$$\begin{aligned}
& z^{\frac{M}{M-N}+1} \varphi^{(3)}(1/z) {}_{(3)}\langle B|\phi_{M+1+i}^*(z) - z^{-\frac{M}{M-N}-1} \varphi^{(3)}(z) {}_{(3)}\langle B|\phi_{M+1+i}(1/z) \\
& = q^{-\frac{M}{M-N}-1} (q - q^{-1})^M \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
& \times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (1 - q/w_k w_{k+1})}{\prod_{k=0}^M D(w_k, w_{k+1})} \prod_{k=1}^M (1 - w_k^2) \\
& \times \frac{(1 + w_{M+1}^2)}{(1 - rq^{L-2M-2}w_L)(1 - q^{2M+K-L+2}w_{L+K}/r)} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1 - qw_{M+k} w_{M+k+1})}{(1 - q^{\epsilon_k} w_{M+k} w_{M+k+1})(1 - q^{\epsilon_k} w_{M+k}/w_{M+k+1})} \\
& \times {}_{(3)}\langle B| e^{Q_{h_1^* - h_1 \dots - h_{M+i} - c_i} h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k))} \\
& \times e^{\sum_{k=1}^{i-1} (c_+^k(qw_{M+k}) + c_+^k(q/w_{M+k})) - \sum_{k=1}^i (c_+^k(q^{1+\epsilon_k} w_{M+k}) + c_+^k(q^{1-\epsilon_k}/w_{M+k}))}.
\end{aligned} \tag{6.66}$$

Here the integration contour \tilde{C}_{M+1+i} encircles $w_k = 0, qw_{k-1}^{\pm 1}$ not but $w_k = q^{-1}w_{k-1}^{\pm 1}$ for $1 \leq k \leq M+i$. Using relation $w_1(1 - q/w_1 w_2) - (w_1 \leftrightarrow w_1^{-1}) = (w_1 - w_1^{-1})$, we symmetrize the variables w_1, w_2, \dots, w_M , iteratively. We have

$$\begin{aligned}
& q^{-\frac{M}{M-N}-1} (q - q^{-1})^M (-1/2)^M \varphi^{(3)}(z) \varphi^{(3)}(1/z) (z - z^{-1}) e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
& \times \sum_{\epsilon_1, \dots, \epsilon_i = \pm} \epsilon_i \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(w_{M+1}^{-1} + w_{M+1})}{(1 - rq^{L-2M-2}w_L)(1 - q^{2M+K-L+2}w_{L+K}/r)} \\
& \times {}_{(3)}\langle B| e^{Q_{h_1^* - h_1 \dots - h_{M+i} - c_i} h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k)) + \sum_{k=1}^{i-1} (c_+^k(qw_{M+k}) + c_+^k(q/w_{M+k}))} \\
& \times w_{M+1} \prod_{k=1}^{i-1} \frac{\epsilon_k q^{\epsilon_k} (1 - qw_{M+k} w_{M+k+1}) e^{-c_+^k(q^{1+\epsilon_k} w_{M+k}) - c_+^k(q^{1-\epsilon_k}/w_{M+k})}}{(1 - q^{\epsilon_k} w_{M+k} w_{M+k+1})(1 - q^{\epsilon_k} w_{M+k}/w_{M+k+1})} e^{-c_+^i(q^{1+\epsilon_i} w_{M+i}) - c_+^i(q^{1-\epsilon_i}/w_{M+i})}.
\end{aligned} \tag{6.67}$$

We use the relation (6.57) for the variables w_{M+1}, \dots, w_{L-1} , iteratively. We use the relation (6.57) for the variable w_L . Using the relation (6.35), we symmetrize the variables $w_{L+1}, \dots, w_{L+K-1}$, iteratively. Using the relation (6.16), we symmetrize the variables $w_{L+K}, \dots, w_{M+i-1}$, iteratively. Then we have

$$\begin{aligned}
& (-r)q^{-\frac{M}{M-N}+K+L-2M-5}(q-q^{-1})^M(-1/2)^M\varphi^{(3)}(z)\varphi^{(3)}(1/z)(z-z^{-1})e^{\frac{\pi\sqrt{-1}}{M-N}M} \\
& \times \prod_{k=1}^{M+i} \int_{\tilde{C}_{M+1+i}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\prod_{k=1}^M (w_k^{-1} - w_k)^2}{\prod_{k=0}^M D(w_k, w_{k+1})} \frac{(w_{M+1}^{-1} + w_{M+1})}{(1 - rq^{L-2M-2}w_L)(1 - rq^{L-2M-2}/w_L)} \\
& \times {}_{(3)}\langle B | e^{Q_{h_1^* - h_1 \dots - h_{M+i} - c_i}} e^{h_+^{*1}(qz) + h_+^{*1}(q/z) - \sum_{k=1}^{M+i} (h_+^k(qw_k) + h_+^k(q/w_k)) + \sum_{k=1}^{i-1} (c_+^k(qw_{M+k}) + c_+^k(q/w_{M+k}))} \\
& \times \prod_{k=1}^{i-1} \frac{(-1 + w_{M+k}^2)e^{-c_+^k(w_{M+k}) - c_+^k(q^2/w_{M+k})}}{(1 - w_{M+k}w_{M+k+1}/q)(1 - qw_{M+k}/w_{M+k+1})} \\
& \times \left(e^{-c_+^i(q^2w_{M+i}) - c_+^i(1/w_{M+i})} - e^{-c_+^i(w_{M+i}) - c_+^i(q^2/w_{M+i})} \right) = 0. \tag{6.68}
\end{aligned}$$

Here we have used

$$\oint_{|w|=1} \frac{dw}{w} e^{c_+^{i-1}(qw) + c_+^{i-1}(q/w)} \left(e^{-c_+^i(q^2w) - c_+^i(1/w)} - e^{-c_+^i(w) - c_+^i(q^2/w)} \right) f(w) = 0,$$

where $f(w) = f(1/w)$.

Q.E.D.

Now we have shown (6.48) for every $j = L + K + 1, \dots, M + N + 2$.

Acknowledgement

This work is supported by the Grant-in-Aid for Scientific Research C (21540228) from Japan Society for Promotion of Science.

A Figure

In this appendix we summarize the figures that we use in the section 2.

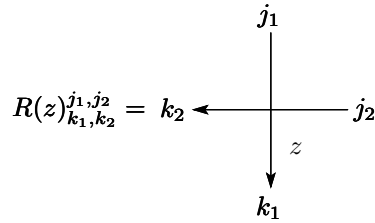


Fig.1. R -matrix

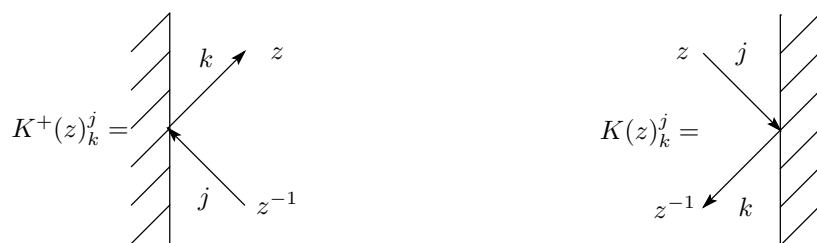


Fig.2. K -matrix

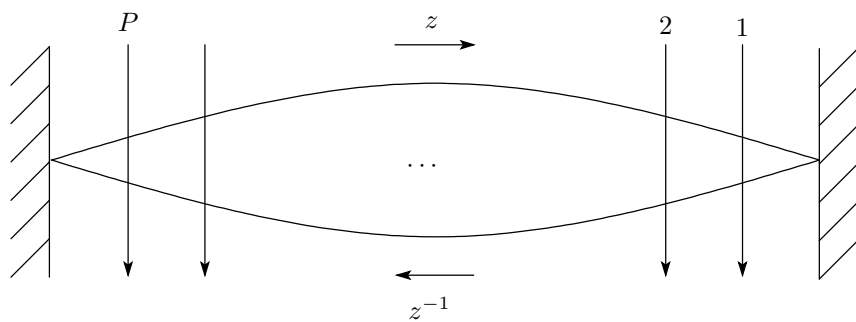


Fig.3. Transfer matrix $T_B^{fin}(z)$

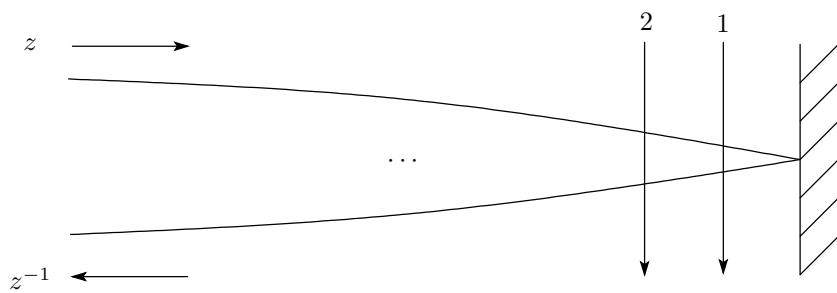


Fig.4. Transfer matrix $\tilde{T}_B^{(i)}(z)$



Fig.5. Vertex operator $\tilde{\Phi}_j(z)$

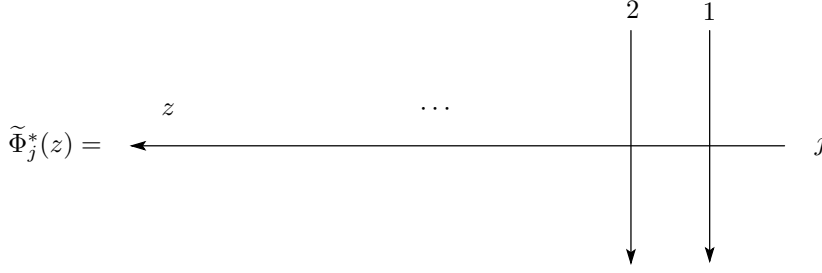


Fig.6. dual Vertex operator $\tilde{\Phi}_j^*(z)$

B K -matrix

In this appendix, we classify diagonal solutions of the boundary Yang-Baxter equation associated with $U_q(\widehat{sl}(M+1|N+1))$. Let us set the vector space $V = \oplus_{j=1}^{M+N+2} \mathbf{C}v_j$. Let us consider the R -matrix $\bar{R}(z) \in \text{End}(V \otimes V)$ introduced in (2.3), (2.4), (2.5) and (2.6). Non-zero elements of the R -matrix are restricted to the followings.

$$\bar{R}(z)_{i,j}^{i,j} \neq 0, \quad \bar{R}(z)_{i,j}^{j,i} \neq 0, \quad (1 \leq i, j \leq M+N+2). \quad (\text{B.1})$$

Let us study the K -matrix $\bar{K}(z) \in \text{End}(V)$ defined as followings.

$$\bar{K}(z) \in \text{End}(V), \quad \bar{K}(z)v_j = \sum_{k=1}^{M+N+2} v_k \bar{K}(z)_k^j, \quad (\text{B.2})$$

where we assume the diagonal matrix.

$$\bar{K}(z)_k^j = \delta_{j,k} \bar{K}(z)_j^j. \quad (\text{B.3})$$

The graded boundary Yang-Baxter equation

$$\bar{K}_2(z_2) \bar{R}_{21}(z_1 z_2) \bar{K}_1(z_1) \bar{R}_{12}(z_1/z_2) = \bar{R}_{21}(z_1/z_2) \bar{K}_1(z_1) \bar{R}_{12}(z_1 z_2) \bar{K}_2(z_2), \quad (\text{B.4})$$

is equivalent to the following two relations for $1 \leq j < k \leq M+N+2$.

$$\bar{R}(z)_{j,k}^{j,k} = \bar{R}(z)_{k,j}^{k,j}, \quad (\text{B.5})$$

$$\begin{aligned} & \bar{R}(z_1/z_2)_{j,k}^{j,k} \left(\bar{R}(z_1 z_2)_{k,j}^{j,k} \bar{K}(z_1)_k^k \bar{K}(z_2)_k^k - \bar{R}(z_1 z_2)_{j,k}^{k,j} \bar{K}(z_1)_j^j \bar{K}(z_2)_j^j \right) \\ & + \bar{R}(z_1 z_2)_{j,k}^{j,k} \left(\bar{R}(z_1/z_2)_{j,k}^{k,j} \bar{K}(z_2)_k^k \bar{K}(z_1)_j^j - \bar{R}(z_1/z_2)_{k,j}^{j,k} \bar{K}(z_1)_k^k \bar{K}(z_2)_j^j \right) = 0. \end{aligned} \quad (\text{B.6})$$

The first condition (B.5) holds for (2.4), (2.5), (2.6). The second condition (B.6) is written as following.

$$\left(1 - \frac{z_1}{z_2} \right) \left(z_1 z_2 - \frac{\bar{K}(z_1)_j^j}{\bar{K}(z_1)_k^k} \frac{\bar{K}(z_2)_j^j}{\bar{K}(z_2)_k^k} \right) + (1 - z_1 z_2) \left(\frac{\bar{K}(z_1)_j^j}{\bar{K}(z_1)_k^k} - \frac{z_1}{z_2} \frac{\bar{K}(z_2)_j^j}{\bar{K}(z_2)_k^k} \right) = 0. \quad (\text{B.7})$$

Differentiating partially (B.7), at $(z_1, z_2) = (z, 1)$, with respect to z_2 , we have the following necessary condition.

$$\frac{\bar{K}(z)_j^j}{\bar{K}(z)_k^k} = \frac{1 - \beta z}{1 - \beta/z} \quad (\beta \in \mathbf{C}). \quad (\text{B.8})$$

This satisfies (B.7) for all $\beta \in \mathbf{C}$. Taking into account of simultaneous compatibility for $1 \leq j < k \leq M + N + 2$, we have the following three kinds of general diagonal solutions of the boundary Yang-Baxter equation associated with $U_q(\widehat{\mathfrak{sl}}(M + 1|N + 1))$.

CASE 1 : One diagonal element. Dirichlet boundary condition.

$$\bar{K}(z)_k^j = \delta_{j,k}. \quad (\text{B.9})$$

CASE 2 : Two different diagonal elements. We assume $1 \leq L \leq M + N + 1$ and $r \in \mathbf{C}$.

$$\bar{K}(z)_k^j = \begin{cases} 1 & (1 \leq j = k \leq L), \\ \frac{1-r/z}{1-rz} & (L+1 \leq j = k \leq M+N+2), \\ 0 & (1 \leq j \neq k \leq M+N+2). \end{cases} \quad (\text{B.10})$$

CASE 3 : Three different diagonal elements. We assume $1 \leq L, 1 \leq K, L + K \leq M + N + 1$ and $r \in \mathbf{C}$.

$$\bar{K}(z)_k^j = \begin{cases} 1 & (1 \leq j = k \leq L), \\ \frac{1-r/z}{1-rz} & (L+1 \leq j = k \leq L+K), \\ z^{-2} & (L+K+1 \leq j = k \leq M+N+2), \\ 0 & (1 \leq j \neq k \leq M+N+2). \end{cases} \quad (\text{B.11})$$

Note. In the earlier studies [10, 11], Case I and Case II have been studied. However Case III is missing in the earlier studies [10, 11]. For instance, we have new solution for $U_q(\widehat{\mathfrak{sl}}(2|1))$, which has three different diagonal elements.

$$\bar{K}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-r/z}{1-rz} & 0 \\ 0 & 0 & z^{-2} \end{pmatrix}. \quad (\text{B.12})$$

C Normal Ordering

In this appendix we summarize the normal orderings. The following normal orderings are convenient to calculations in a proof of main theorem.

$$e^{h_+^{*1}(z)} e^{-h_-^1(w)} = \frac{1}{(1-qw/z)} ::, \quad (\text{C.1})$$

$$e^{-h_+^1(w)} e^{h_-^{*1}(z)} = \frac{1}{(1-qz/w)} ::, \quad (\text{C.2})$$

$$e^{h_+^{*1}(z)} e^{-h_-^j(w)} = 1 :: \quad (2 \leq j \leq M+N+1), \quad (\text{C.3})$$

$$e^{-h_+^j(w)} e^{h_-^{*1}(z)} = 1 :: \quad (2 \leq j \leq M+N+1), \quad (\text{C.4})$$

$$e^{-h_+^j(w_1)} e^{-h_-^{j+1}(w_2)} = \frac{1}{(1-qw_2/w_1)} :: \quad (1 \leq j \leq M), \quad (\text{C.5})$$

$$e^{-h_+^{j+1}(w_1)} e^{-h_-^j(w_2)} = \frac{1}{(1-qw_2/w_1)} :: \quad (1 \leq j \leq M), \quad (\text{C.6})$$

$$e^{-h_+^{M+j}(w_1)} e^{-h_-^{M+j+1}(w_2)} = (1-qw_2/w_1) :: \quad (1 \leq j \leq M), \quad (\text{C.7})$$

$$e^{-h_+^{M+j+1}(w_1)} e^{-h_-^{M+j}(w_2)} = (1-qw_2/w_1) :: \quad (1 \leq j \leq N). \quad (\text{C.8})$$

The following normal orderings are convenient to get the integral representations of the vertex operators.

$$\phi_1^*(z)X_1^-(qw) = e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qz(1-qw/z)} ::, \quad (\text{C.9})$$

$$X_1^-(qw)\phi_1^*(z) = -e^{\frac{\pi\sqrt{-1}}{M}} \frac{1}{qw(1-qz/w)} ::, \quad (\text{C.10})$$

$$\phi_1^*(z)X_j^-(w) = 1 :: \quad (2 \leq j \leq M), \quad (\text{C.11})$$

$$X_j^-(w)\phi_1^*(z) = 1 :: \quad (2 \leq j \leq M), \quad (\text{C.12})$$

$$\phi_1^*(z)X_{M+1,\epsilon}^-(w) = -1 :: \quad (\epsilon = \pm), \quad (\text{C.13})$$

$$X_{M+1,\epsilon}^-(w)\phi_1^*(z) = 1 :: \quad (\epsilon = \pm), \quad (\text{C.14})$$

$$X_j^-(qw_1)X_{j+1}^-(qw_2) = \frac{1}{qw_1(1-qw_2/w_1)} :: \quad (1 \leq j \leq M), \quad (\text{C.15})$$

$$X_{j+1}^-(qw_1)X_j^-(qw_2) = \frac{-1}{qw_1(1-qw_2/w_1)} :: \quad (1 \leq j \leq M), \quad (\text{C.16})$$

$$X_j^-(qw_1)X_k^-(qw_2) = 1 :: \quad (|j-k| \geq 2), \quad (\text{C.17})$$

$$X_{M+j,+}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) = \frac{(1-qw_2/w_1)}{q(1-w_2/qw_1)} :: \quad (\epsilon = \pm), \quad (\text{C.18})$$

$$X_{M+1+j,+}^-(w_1)X_{M+j,\epsilon}^-(w_2) = 1 :: \quad (\epsilon = \pm), \quad (\text{C.19})$$

$$X_{M+j,-}^-(w_1)X_{M+j+1,\epsilon}^-(w_2) = q :: \quad (\epsilon = \pm), \quad (\text{C.20})$$

$$X_{M+j+1,-}^-(w_1)X_{M+j,\epsilon}^-(w_2) = \frac{(1-qw_1/w_2)}{(1-w_1/qw_2)} :: \quad (\epsilon = \pm). \quad (\text{C.21})$$

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